Options on Leveraged ETF: Calibrations and Error Analysis

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What is a Leveraged ETF?

A leveraged ETF is an asset that gives a multiple of the daily return of another asset. If \( s_t \) is the asset value and \( l_t \) a LETF with multiple \( m \) then we have

\[
\frac{dl_t}{l_t} = m \frac{ds_t}{s_t} + (1 - m)rdt
\]

and if we specify the dynamic for \( s_t \) to be of Heston type

\[
\begin{align*}
\frac{ds_t}{s_t} &= (r - q)dt + \sqrt{v_t}d\omega_t^1, \\
v_t &= \kappa(\theta - v_t)dt + \sigma \sqrt{v_t}d\omega_t^2,
\end{align*}
\]

with \((w_t^1, w_t^2)_{t \geq 0}\) a two-dimensional Brownian motion with correlation structure \(d\langle w_1, w_2 \rangle_t = \rho dt\) then \( l_t \) is known. We have

\[
d\ln l_t = md\ln s_t + \frac{m - m^2}{2}v_tdt + (1 - m)rdt,
\]

For \( m \in \{-3, -2, -1, 1, 2, 3\} \) bias is negative.
What is a Leveraged ETF?

\[ \ln l_t - \ln l_0 = m(\ln s_t - \ln s_0) + \frac{m-m^2}{2} \int_0^t v_u du + (1-m)rt \]

The bias depends on the integrated volatility and can be substantial. Also we rewrite as

\[ l_t = l_0 \left( \frac{s_t}{s_0} \right)^m e^{\left( \frac{m-m^2}{2} \right) \int_0^t v_u du + (1-m)rt} \]

It is similar to the outcome of a CPPI of Black&Perold.

LETF are written on many different assets (Index, Fx, commodities) but by far the most important ones are those on the SP500.
# Leveraged ETF on the S&P500

<table>
<thead>
<tr>
<th>Fund Name</th>
<th>Ticker Name</th>
<th>Leverage Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proshares UltraPro Short S&amp;P500 ETF</td>
<td>SPXU</td>
<td>-3</td>
</tr>
<tr>
<td>Proshares UltraShort S&amp;P500 ETF</td>
<td>SDS</td>
<td>-2</td>
</tr>
<tr>
<td>Proshares Short S&amp;P500 ETF</td>
<td>SH</td>
<td>-1</td>
</tr>
<tr>
<td>SPDR S&amp;P500 ETF</td>
<td>SPY</td>
<td>+1</td>
</tr>
<tr>
<td>Proshares Ultra S&amp;P500 ETF</td>
<td>SSO</td>
<td>+2</td>
</tr>
<tr>
<td>Proshares UltraPro S&amp;P500 ETF</td>
<td>UPRO</td>
<td>+3</td>
</tr>
</tbody>
</table>
Leveraged ETF on the S&P500

Note. Times series for LETF from 2011/06/01 to 2012/06/01.
Pricing Options on Leveraged ETF

Option on $l_t$ are available and their price is:

$$c(t, l_0, v_0) = e^{-rt}E[(l_t - K)_+]$$

$$= e^{-rt}E\left[ \left( l_0 \left( \frac{s_t}{s_0} \right)^m e^{\left( \frac{m-m^2}{2} \right) \int_0^t v_u du + (1-m)rt} - K \right)_+ \right]$$

It is like a Power option with a volatility bias component. Also, it an hybrid product:

- Vanilla options $\rightarrow$ "pure" stock derivative product.
- VIX options $\rightarrow$ "pure" volatility derivative product.
- LETF options $\rightarrow$ stock/volatility derivative product.
Pricing Options on Leveraged ETF

\[
c(t, l_0, v_0) = e^{-rt} \mathbb{E} [(l_t - K)_+]
\]
\[
= e^{-rt} \mathbb{E} \left[ \left( l_0 \left( \frac{s_t}{s_0} \right)^m e^{-\left( m-m_0^2 \right) \int_0^t v_u du + (1-m)rt} - K \right) \right]_+
\]
\[
= e^{-rt} \mathbb{E} [(l_0 e^{x_t} - K)_+]
\]
\[
= e^{-rt} \int_{-\infty}^{+}\infty (l_0 e^{x} - K)_+ f(x) dx,
\]

where \( f(x) \) is the density of \( x_t = m \ln \left( \frac{s_t}{s_0} \right) + \left( \frac{m-m_0^2}{2} \right) \int_0^t v_u du + (1-m)rt \). We denote by \( \varphi(t, z) = \mathbb{E} [e^{izx_t}] \) the characteristic function of \( x_t \), and we have

\[
c(t, l_0, v_0) = \frac{e^{-rt}}{2\pi} \int_{-\infty+i\gamma}^{+\infty+i\gamma} \varphi(t, z) \int_{-\infty}^{+\infty} (l_0 e^{x} - K)_+ e^{-izx} dx dz
\]
\[
= \frac{e^{-rt}}{2\pi} \int_{-\infty+i\gamma}^{+\infty+i\gamma} \varphi(t, z) \frac{K^{1-iz}l_0^{iz}}{iz(iz-1)} dz,
\]

where \( \gamma = \Im(z) < -1 \). Letting \( k_0 = \ln \left( \frac{K}{l_0} \right) \), the above equation can be simplified to

\[
c(t, l_0, v_0) = \frac{Ke^{-rt}}{\pi} \int_{0+i\gamma}^{+\infty+i\gamma} e^{-ik_0z} \varphi(t, z) \frac{z}{iz(iz-1)} dz.
\]
Pricing Options on Leveraged ETF

The characteristic function of \((x_t)_{t \geq 0}\) is linked to the moment generating function of the stock \(s_t\) as follows

\[
\varphi(t, z) = \mathbb{E}[e^{izx_t}] = \mathbb{E}\left[e^{iz(m \ln \frac{s_t}{s_0} + \frac{m-m^2}{2} \int_0^t v_udu + (1-m)rt)}\right] = e^{i(1-m)zrt} G\left(t, izm, iz \frac{m-m^2}{2}, 0, v_0\right)
\]

with

\[
G(t, z_1, z_2, \ln s_0, v_0) = \mathbb{E}[e^{z_1 \ln s_t + z_2 \int_0^t v_udu}] = e^{z_1 \ln s_0 + a(t) + b(t)v_0}
\]

where

\[
a(t) = \frac{2\kappa \theta}{\sigma^2} \left(t\lambda_- - \ln \left(\frac{\lambda_+ - \lambda_- e^{-\sqrt{\Delta}t}}{\lambda_+ - \lambda_-} \right)\right) + (r-q)z_1 t,
\]

\[
b(t) = (z_1^2 - z_1 + 2z_2) \frac{1 - e^{-\sqrt{\Delta}t}}{\lambda_+ - \lambda_- e^{-\sqrt{\Delta}t}},
\]

\[
\lambda_+ = \frac{(\kappa - z_1 \rho \sigma) \pm \sqrt{\Delta}}{2},
\]

\[
\Delta = (\kappa - z_1 \rho \sigma)^2 - \sigma^2(z_1^2 - z_1 + 2z_2).
\]
Pricing Options on Leveraged ETF

As a result

\[ c(t, l_0, v_0) = \frac{K e^{-rt}}{\pi} \int_{0+i\gamma}^{+\infty+i\gamma} e^{-izk_0} \varphi(t, z) \frac{\varphi(t, z)}{iz(iz - 1)} dz. \]

can be computed using FFT (as a vanilla option).
Options on Leveraged ETF

SDS(-2)
Options on Leveraged ETF

SH(-1)
Options on Leveraged ETF

SPY(+1)
Options on Leveraged ETF

SSO(+2)
Calibration of options on Leveraged ETF

For each leverage ratio we calibrate the options by solving the following optimization problem for the day 2011/10/24:

$$\min_{\nu_0, \kappa, \theta, \sigma, \rho} \frac{1}{N} \sum_{i}^{N} \left( \sigma^{market}_{imp}(t_i, K_i, m) - \sigma^{model}_{imp}(t_i, K_i, m) \right)^2$$

<table>
<thead>
<tr>
<th></th>
<th>$\nu_0$</th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>ErrorVol</th>
<th>Error Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPXU</td>
<td>0.0253</td>
<td>2.6308</td>
<td>0.1438</td>
<td>0.4282</td>
<td>-0.664</td>
<td>$3.949 \times 10^{-2}$</td>
<td>$6.093 \times 10^{-4}$</td>
</tr>
<tr>
<td>SDS</td>
<td>0.0823</td>
<td>8.3064</td>
<td>0.0781</td>
<td>1.6757</td>
<td>-0.535</td>
<td>$2.108 \times 10^{-3}$</td>
<td>$3.891 \times 10^{-5}$</td>
</tr>
<tr>
<td>SH</td>
<td>0.0708</td>
<td>2.7108</td>
<td>0.0948</td>
<td>0.9343</td>
<td>-0.320</td>
<td>$5.835 \times 10^{-4}$</td>
<td>$5.152 \times 10^{-6}$</td>
</tr>
<tr>
<td>SPY</td>
<td>0.0854</td>
<td>2.4816</td>
<td>0.1345</td>
<td>1.6613</td>
<td>-0.739</td>
<td>$4.654 \times 10^{-4}$</td>
<td>$2.13 \times 10^{-6}$</td>
</tr>
<tr>
<td>SSO</td>
<td>0.0698</td>
<td>3.7320</td>
<td>0.0983</td>
<td>1.3296</td>
<td>-0.638</td>
<td>$2.505 \times 10^{-3}$</td>
<td>$3.697 \times 10^{-5}$</td>
</tr>
<tr>
<td>UPRO</td>
<td>0.0612</td>
<td>0.5308</td>
<td>0.1549</td>
<td>0.4747</td>
<td>-0.649</td>
<td>$3.765 \times 10^{-3}$</td>
<td>$3.946 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

- Calibration errors smaller for -1 and +1.
- Large leverage ratio (absolute value) implies large error.
- The ratios lead to same kind of parameters.
- SH leads to different $\rho \rightarrow$ explains why not used in Ahn et al.?
- Feller condition not satisfied by most of parameter sets.
Options on Leveraged ETF: Open problems

We know that there is a moment explosion problem: for $z > 1$ we can have $t^*$ s.t.

$$\mathbb{E}[s_{t^*}^z] = +\infty$$

(1)

The fact that $\rho < 0$ helps but

1. Large $m$ requires higher moments, are they well defined?
2. Here we need to consider negative moments! not studied so far...
Conclusions

- We perform a calibration of options on LETF related to the SP500
- We find that higher leverage ratio leads to larger calibration errors
- We obtain similar parameter sets
- We calibrate all the parameters
- We raise some open problems