The $\alpha$-Hypergeometric Stochastic Volatility Model

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The Positivity Problem in Finance (and other fields)

Two simple ways to have positivity

\[ x^2 \quad e^x \]

Positivity is important in finance for:

- Volatility.
- Interest rates.
- Stock price.

and Noise is given by the Gaussian distribution, hence in $\mathbb{R}$. 
Positivity in Econometrics

The GARCH:

\[ r_t = \sigma_t \epsilon_t \]
\[ \sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 + \beta_1 \epsilon_{t-1}^2 \]

The EGARCH:

\[ r_t = \sigma_t \epsilon_t \]
\[ \ln \sigma_t^2 = \alpha_0 + \alpha_1 g(\epsilon_{t-1}) + \beta_1 \ln \sigma_{t-1}^2 \]
Positivity in Interest rates

Zero coupon bond

\[ B(t, T) = \mathbb{E}_t^Q \left[ e^{-\int_t^T r_u du} \right] \]

Vasicek (Ornstein-Ulhenbeck):

\[ dr_t = \kappa(\theta - r_t)dt + \sigma dr_t \]

easy but Gaussian!

Dothan:

\[ dr_t = \kappa r_t dt + \sigma r_t dw_t \]

positive but much more complicated.
Equity Derivatives

For the stochastic volatility models:

\[ ds_t = s_t \sigma_t dw_1^t \]  

and

\[ d\sigma_t = a \sigma_t dt + b \sigma_t dw_2^t \]  
\[ d\ln \sigma_t = a(b - \ln \sigma_t) dt + \alpha dw_2^t \]  
\[ d\sigma_t = a(b - \sigma_t) dt + \alpha dw_2^t \]

- Hull & White (2): volatility non stationary but exponential so positive!
- Chesney & Scott (3): logarithm of volatility Ornstein-Ulhenbeck so Gaussian but volatility is exponential so positive!
- Stein & Stein (4): volatility is Ornstein-Ulhenbeck so Gaussian, volatility is negative.

but

- Hull & White not good because volatility is a geometric Brownian motion.
- Chesney & Scott, we don't know the stock density or its characteristic function. Cannot calibrate the model.
- Stein & Stein (4), we don't know the stock characteristic function (option pricing by FFT) but volatility is Gaussian!
Equity Derivatives

\[ ds_t = s_t \sqrt{\sigma_t} d\omega^1_t \]

and

\[ d\sigma_t = a(b - \sigma_t) dt + \alpha \sqrt{\sigma_t} d\omega^2_t \] (5)

- The volatility is positive and we know the characteristic function of the stock.
- The Feller condition \(2ab > \alpha^2\) ensures that \(\sigma_t > 0\).

Option contains integrated volatility

\[ E_t^Q \left[ \left( s_t e^{\frac{1}{2} \int_t^T \sigma_u du} + \int_t^T \sigma_u d\omega^1_u - K \right)_+ \right] \]

Whether the volatility oscillates a lot (large \( \alpha \)) or not (small \( \alpha \)) option convey little (no) information on that aspect.
Equity Derivatives

The Feller condition is not satisfied in practice:

1. The volatility can touch 0.
2. The volatility distribution is too close to 0.

In fact the square root process is positivity using the $x^2$ function.

Positivity using $e^x$ doesn’t work but the exponentiation is appealing.
The Hypergeometric Stochastic Volatility Model

The forward price dynamic:

\[ df_t = f_t e^{vt} dw_{1,t} \]  \hspace{1cm} (6)

\[ dv_t = (a - be^{\alpha vt}) dt + \sigma dw_{2,t} \]  \hspace{1cm} (7)

with \( dw_{1,t} dw_{2,t} = \rho dt \) (controls the leverage).

- Volatility \( v_t \) looks like an OU process.
- Stock volatility \( e^{vt} \) is positive by construction.

For \( \alpha = 1 \) we know how to compute the Mellin transform of the stock (so option pricing is possible).
The Hypergeometric Stochastic Volatility Model

\[
\mathbb{E} \left[ \left( \frac{f_t}{f_0} \right)^\lambda \right] = \mathbb{E} \left[ \exp \left( -\frac{\lambda}{2} \int_0^t e^{2\nu_u} du + \lambda \int_0^t e^{2\nu_u} dw_{1,u} \right) \right] \\
= e^{-\frac{\lambda \rho}{\sigma} e^{\nu_0}} \mathbb{E} \left[ \exp \left( \alpha_0 e^{\nu_t} + \alpha_1 \int_0^t e^{\nu_s} ds - \frac{\alpha_2^2}{2} \int_0^t e^{2\nu_s} ds \right) \right]
\]

with

\[
\alpha_0 = \frac{\lambda \rho}{\sigma} \quad \alpha_1 = -\frac{\lambda \rho}{\sigma} \left( a + \frac{\sigma^2}{2} \right) \quad \alpha_2 = -\lambda^2 (1 - \rho^2) - \frac{2b\rho \lambda}{\sigma} + \lambda.
\]

and \( dv_t = (a - be^{\nu_t}) dt + \sigma dw_{2,t} \).

Girsanov's theorem to cancel the drift of the volatility

\[
\mathbb{E} \left[ \left( \frac{f_t}{f_0} \right)^\lambda \right] = e^{-\frac{a}{\sigma^2} e^{\nu_0} + \frac{b - \lambda \rho}{\sigma^2} e^{\nu_0} - \frac{a^2}{2\sigma^2} \mathbb{E}^{Q} \left[ \exp \left( a \nu_t + \beta_0 e^{\nu_t} + \beta_1 \int_0^t e^{\nu_s} ds - \frac{\beta_2^2}{2} \int_0^t e^{2\nu_s} ds \right) \right]} \\
= e^{-\frac{a}{\sigma^2} e^{\nu_0} + \frac{b - \lambda \rho}{\sigma^2} e^{\nu_0}} \mathbb{E}^{Q} \left[ \exp \left( a \nu_t + \beta_0 e^{\nu_t} + \beta_1 \int_0^t e^{\nu_s} ds - \frac{\beta_2^2}{2} \int_0^t e^{2\nu_s} ds \right) \right]
\]

with

\[
\beta_0 = \frac{\lambda \rho \sigma - b}{\sigma^2} \quad \beta_1 = (b - \lambda \rho \sigma) \left( \frac{a}{\sigma^2} + \frac{1}{2} \right) \quad \beta_2 = -\lambda^2 (1 - \rho^2) + \lambda \left( 1 - \frac{2b \rho}{\sigma} \right) + \frac{b^2}{\sigma^2}.
\]

and \( dv_t = \sigma d\tilde{w}_{2,t} \)
The Hypergeometric Stochastic Volatility Model

\[ F(t, v) = E^Q \left[ \exp \left( \frac{av_t}{\sigma^2} + \beta_0 e^{v_t} + \beta_1 \int_0^t e^{v_s} ds - \frac{\beta_2^2}{2} \int_0^t e^{2v_s} ds \right) \right] \]  

(8)

and \( F(0, v) = \exp \left( \frac{av}{\sigma^2} + \beta_0 e^v \right) \). \( F(t, v) \) solves the PDE:

\[
\partial_t F = \frac{\sigma^2}{2} \frac{d^2 F}{dv^2} - \frac{\beta_2^2}{2} e^{2v} F + \beta_1 e^v F,
\]

so \( F(t) = e^{-Ht} F(0) \) and in integral form:

\[
F(t, v_0) = \int_{-\infty}^{+\infty} q(\sigma^2 t, v_0, y) F(0, y) dy
\]

- \( q \) is the heat kernel.

- \( -\frac{\beta_2^2}{2} e^{2v} + \beta_1 e^v \) is the potential (well known): Morse potential.
The Hypergeometric Stochastic Volatility Model

The Laplace transform of the HK is known

\[
G(v, y; s^2/2) = \int_0^{+\infty} e^{-\frac{s^2}{2}t} q(t, v, y) dt = \int_0^{+\infty} e^{-\frac{s^2}{2}t} e^{-Ht} dt.
\]

\[
= \left(\frac{s^2}{2} + H\right)^{-1}
\]

\(G\) is the fundamental solution (the Green function, or the resolvant) of \(H + \frac{s^2}{2} = 0\) that is to say \(G\) solves:

\[
-\frac{\sigma^2}{2} \frac{d^2 G}{dv^2} + \frac{\beta_2^2}{2} e^{2v} G - \beta_1 e^v G + \frac{s^2}{2} = \delta_y
\]  

(9)

\[
G(v, y; \eta^2/2) = \frac{\Gamma\left(\eta - \frac{\nu_1}{\nu_2} + \frac{1}{2}\right)}{\nu_2 \Gamma(1 + 2\eta)} e^{-(v+y)/2} W_{\frac{\nu_1}{\nu_2}, \eta} (2\nu_2 e^{y>}) M_{\frac{\nu_1}{\nu_2}, \eta} (2\nu_2 e^{y<})
\]

with \(\nu_1, \nu_2\) related to \(\beta_1, \beta_2, \eta\) to \(s\) and \(y_> = \max(v, y), y_< = \min(v, y)\), \(W_{\kappa, \eta}\) and \(M_{\kappa, \eta}\) are the Whittaker functions (related to confluent hypergeometric functions):

\[
W_{\kappa, \eta}(z) = z^{\eta+\frac{1}{2}} e^{-z/2} \Psi\left(\eta - \kappa + \frac{1}{2}, 1 + 2\eta; z\right)
\]

\[
M_{\kappa, \eta}(z) = z^{\eta+\frac{1}{2}} e^{-z/2} \Phi\left(\eta - \kappa + \frac{1}{2}, 1 + 2\eta; z\right).
\]
The Hypergeometric Stochastic Volatility Model

1. \( G \) is known.

2. \( q \) is the inverse Laplace transform of \( G \).

3. We integrate \( q \) over \( F(0, v) \) it gives the Mellin transform of the spot.

4. We compute the inverse Mellin transform of the spot to get the option price.
Conclusions

• we develop a stochastic volatility model with positive volatility
• we provide the main results to perform option pricing
Open Problems

- all the problems are open....