The accurate computation of key properties of Markov and semi-Markov Processes

Jeffrey J Hunter
Auckland University of Technology
New Zealand
Outline

1. Introduction
2. Stationary distributions of Markov chains
3. Mean first passage times
4. Solving for Stationary Distributions
5. Solving for Mean First Passage Times
6. GTH Algorithm
7. MFPT via the Extended GTH Algorithm
8. The Test Problems
9. Computation comparisons for MFPT
1. Introduction

Let $P = [p_{ij}]$ be the transition matrix of an irreducible, discrete time Markov chain (MC) $\{X_n\} (n \geq 0)$ with finite state space $S = \{1, 2, \ldots, N\}$.

\[ i.e. \quad p_{ij} = P\{X_n = j | X_{n-1} = i\} \text{ for all } i, j \in S. \]

We are interested in developing accurate ways of finding two key properties of such chains:

(i) the stationary probabilities $\{\pi_j\}, (1 \leq j \leq N)$.

(ii) the mean first passage times $\{m_{ij}\}, (1 \leq i, j \leq N)$. 
2. Stationary distributions

Let \( \pi^T = (\pi_1, \pi_2, \ldots, \pi_N) \) be the stationary prob. vector of the Markov chain with transition matrix \( P = [p_{ij}] \).

We need to solve \( \pi_j = \sum_{i=1}^{N} \pi_i p_{ij} \) with \( \sum_{i=1}^{N} \pi_i = 1 \), i.e.

\[
\pi^T (I - P) = 0^T \text{ with } \pi^T e = 1.
\]
3. Mean first passage times

Let $T_{ij}$ be the first passage time RV from state $i$ to state $j$, i.e. $T_{ij} = \min \{ n \geq 1 \text{ such that } X_n = j \text{ given that } X_0 = i \}$. $T_{ii}$ is the first return to state $i$.

Let $m_{ij} = E[T_{ij} \mid X_0 = i]$, the mean first passage time from state $i$ to state $j$. 
The mean first passage times

Let $M = [m_{ij}]$ be the matrix of mean first passage times.

It is well known that

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj},$$

with $m_{jj} = 1/\pi_j$.

$M$ satisfies the matrix equation

$$(I - P)M = E - PD,$$

where $E = [1] = ee^T$, and

$D = M_d = [\delta_{ij} m_{ij}] = (\Pi_d)^{-1}$ (with $\Pi = e\pi^T$).
4. Solving for the stationary distribution

If \( G = [I - P + tu^T]^{-1} \) where \( u, t \) such that \( u^T e \neq 0 \), \( \pi^T t \neq 0 \),

\[
\pi^T = \frac{u^T G}{u^T Ge}.
\]

(Paige, Styan, Wachter, 1975), (Kemeny, 1981), (Hunter, 1982)

In particular if \( G = [I - P + eu^T]^{-1} \) then \( \pi^T = u^T G \)
5. Solving for mean first passage times

(i) If $G$ is any g-inverse of $I - P$, then

$$M = [G\Pi - E(G\Pi)_d + I - G + EG_d]D.$$  \quad \text{(Hunter, 1982)}$$

(ii) If $Ge = ge$ for some $g$

$$\Leftrightarrow M = [I - G + EG_d]D.$$ \quad \text{(Hunter, 2013)}$$

The "standard algorithm" is $M = [I - Z + EZ_d]D$ where $Z = [I - P + e\pi^T]^{-1}$, Kemeny and Snell's "fundamental matrix"
Solving for mean first passage times

Hunter (2007) presented a "simple algorithm" which is the simplest method to simultaneously compute the stationary distribution and the MFPTs.

If \( G_{eb} = [I - P + ee_b^T]^{-1} = \begin{bmatrix} g_{ij} \end{bmatrix} \),

then \( \pi_j = g_{bj}, \quad j = 1, 2, \ldots, N, \)

and \( m_{ij} = \begin{cases} 1/ g_{bj}, & i = j, \\ (g_{jj} - g_{ij})/ g_{bj}, & i \neq j. \end{cases} \)
Hunter (2014) developed a variety "perturbation algorithms" where the transition matrix is successively updated row by row from an initial simple transition matrix to end up with the required transition matrix. One such algorithm is the following:

(i) Let $K_0 = I$.

(ii) For $i = 1, 2, ..., N$, let $p_i^T = e_i^T P$, $b_i^T = p_i^T - e^T / N$,

$$K_i = K_{i-1}(I + C_i),$$

where $k_i = 1 - b_{i}^T K_{i-1} e_i$ and $C_i = \frac{1}{k_i} e_i b_{i}^T K_{i-1}$.

(iii) At $i = N$, let $K = K_N$ then $\pi^T = \frac{1}{N} e^T K$,

$$M = [I - K + EK_d]D,$$

where $D = (\Pi_d)^{-1}$. 
6. The GTH Algorithm

Let \( P_N = \begin{bmatrix} p_{ij} \end{bmatrix} = \begin{bmatrix} p_{ij}^{(N)} \end{bmatrix} \) be the \( N \times N \) transition matrix associated with a M.C. \( \{X_k, k \geq 0\} \) with state space \( S_N = \{1,2,...,N\} \), and transition probabilities \( p_{ij}^{(N)} = P\{X_{k+1} = j \mid X_k = i\} \).

The general approach is to start with an \( N \)-state Markov chain and reduce the state space by one state at each stage. Thus in stages
\[
S_N = S_{N-1} \cup \{N\}, \quad S_{N-1} = S_{N-2} \cup \{N - 1\},..., \quad S_2 = \{1,2\}.
\]

Once we get to two states we expand the state space one state at a time until we return to the final set \( S_N \).
Assume that the initial M.C. with state space $S_N$ is irreducible and that stationary probability vector is given by $\pi^T = (\pi_1, \pi_2, \ldots, \pi_{N-1}, \pi_N)$

Let $\pi^T = \pi^{(N)T} = (\pi_1^{(N)}, \pi_2^{(N)}, \ldots, \pi_{N-1}^{(N)}, \pi_N^{(N)})$.

From the stationary equations $\pi^{(N)T} = \pi^{(N)T} P_N$ or in element form $\pi_j^{(N)} = \sum_{i=1}^{N} \pi_i^{(N)} p_{ij}^{(N)}$ ($j = 1, 2, \ldots, N$)

equations.

express $\pi_N^{(N)}$ in terms of $\pi_1^{(N)}, \ldots, \pi_{N-1}^{(N)}$:

$$\pi_N^{(N)} = \frac{\sum_{i=1}^{N-1} \pi_i^{(N)} p_{iN}^{(N)}}{\sum_{j=1}^{N-1} p_{Nj}^{(N)}}$$

and eliminate $\pi_N^{(N)}$ from the stationary equations.
Partition the stationary probability vector \( \pi^{(N)} = (\nu^{(N-1)^T}, \pi_N) \) where \( \nu^{(N-1)^T} = (\pi_1^{(N-1)}, \pi_2^{(N-1)}, \ldots, \pi_{N-1}^{(N-1)}) \).

It is easily shown that
\[
\nu^{(N-1)^T} (I_{N-1} - P_{N-1}) = 0^T, \quad \text{where} \quad P_{N-1} = Q_{N-1}^{(N)} - \frac{p_{(N)(c)}^{(N)} p_{(N)(r)}^{(N)} e^{(N)}}{p_{N-1}^{(N)(r)} e^{(N)}}.
\]

Let \( P_{N-1} = \begin{bmatrix} p_{(N-1)}^{i} \\ p_{(N)}^{j} \end{bmatrix} \) then
\[
p_{ij}^{(N-1)} = p_{ij}^{(N)} + \frac{p_{ij}^{(N)} p_{ij}^{(n)}}{S(N)},
\]

where
\[
1 \leq i \leq N - 1, \quad 1 \leq j \leq N - 1.
\]

Note that calculation of the \( S(N) \) and the \( p_{ij}^{(N-1)} \) do not involve subtractions.
Observe

$P_{N-1}$ is a stochastic matrix with state space $S_{N-1}$
$P_{N-1}$ is irreducible
$\nu^{(N-1)T}$ is a scaled stationary prob vector of this $N - 1$ state MC

\[ \pi^{(N-1)T} = (\pi_1^{(N-1)}, \pi_2^{(N-1)}, \ldots, \pi_{N-1}^{(N-1)}) \equiv \frac{1}{1 - \pi_N^{(N)}} \nu^{(N-1)T} \]

so that the first $N - 1$ stationary probs of the $N$-state MC are scaled versions of the $N - 1$ state MC.
We can repeat this process reducing the state space from \( n \) to \( n-1 \) \((n= N, N-1, \ldots, 2)\) with the resulting MC having a stationary distribution that is a scaled version of the first \( n-1 \) components of the stationary distribution of the MC with \( n \) states.

Thus if \( P_n = \begin{bmatrix} p_{ij}^{(n)} \end{bmatrix} \) with \( P_{n-1} = \begin{bmatrix} p_{ij}^{(n-1)} \end{bmatrix} \) then

\[
p_{ij}^{(n-1)} = p_{ij}^{(n)} + \frac{p_{in}^{(n)} p_{nj}^{(n)}}{S(n)}, 1 \leq i \leq n-1, 1 \leq j \leq n-1;
\]

where \( S(n) = 1 - p_{nn}^{(n)} = \sum_{j=1}^{n-1} p_{nj}^{(n)}. \)
Interpretation of the transition probabilities

\[ p_{ij}^{(n-1)} = p_{ij}^{(n)} + \frac{p_{in}^{(n)}p_{nj}^{(n)}}{S(n)}, 1 \leq i \leq n-1, 1 \leq j \leq n-1. \]

The \( p_{ij}^{(n-1)} \) can be interpreted as the transition probability from \( i \) to \( j \) of the M.C. on \( S_n \) restricted to \( S_{n-1} \).

For \((i,j) \in S_{n-1} \times S_{n-1}\) it is possible to jump directly from \( i \) to \( j \) with probability \( p_{ij}^{(n)} \). Alternatively jump from \( i \) to \( j \) via state \( n \), being held at state \( n \) for \( t (= 0, 1, 2, \ldots) \) steps, followed by a jump to \( j \), with probability

\[ p_{in}^{(n)} \left( \sum_{t=0}^{\infty} \left( p_{nn}^{(n)} \right)^t \right) p_{nj}^{(n)} = \frac{p_{in}^{(n)}p_{nj}^{(n)}}{1 - p_{nn}^{(n)}} = \frac{p_{in}^{(n)}p_{nj}^{(n)}}{S(n)}, \]

leading to the general expression for \( p_{ij}^{(n-1)} \).
Since the original M.C. is irreducible (i.e. every state can be reached from every other state) the restricted M.C. must also be irreducible and further since $p_{nn}^{(n)} < 1$, $S(n) > 0$.

If we start with

$$\pi^{(N)^T} = (\pi_1^{(N)}, \pi_2^{(N)}, \ldots, \pi_{N-1}^{(N)}, \pi_N^{(N)}) \equiv (\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n)$$

then the $N - 1$ elements of $\pi^{(N-1)^T}$ are scaled elements of the first $N - 1$ elements of $\pi^{(N)^T}$ and hence of $\pi_1, \pi_2, \ldots, \pi_{n-1}$. Thus each $\pi^{(n)^T}$ is a scaled version of $(\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n)$.

The process continues to $n = 2$, where we have

$$P_2 = \begin{bmatrix}
p_{11}^{(2)} & p_{12}^{(2)} \\
p_{21}^{(2)} & p_{22}^{(2)}
\end{bmatrix}$$

which is a stochastic matrix.
The stationary distribution of this MC will be a scaled version of \( \pi^{(2)T} = (\pi_1^{(2)}, \pi_2^{(2)}) \) or of \((\pi_1, \pi_2)\).

The second stationary equation is \( \pi_2 = \pi_1 p_{12}^{(2)} + \pi_2 p_{22}^{(2)} \) implying

\[
\pi_2 = \pi_1 \frac{p_{12}^{(2)}}{S(2)}.
\]

Note that \( S(2) = 1 - p_{22}^{(2)} = \sum_{j=1}^{1} p_{2j}^{(2)} = p_{21}^{(2)} = p_{1}^{(2)(r)} e^{(1)} \).

We now proceed with increasing the state space.

\[
\pi_3 = \frac{\sum_{i=1}^{2} \pi_i p_{i3}^{(3)}}{\sum_{i=1}^{2} p_{3i}^{(3)}} = \pi_1 \frac{p_{13}^{(3)}}{S(3)} + \pi_2 \frac{p_{23}^{(3)}}{S(3)},
\]

In general, \( \pi_n = \frac{\sum_{i=1}^{n-1} \pi_i p_{in}^{(n)}}{\sum_{i=1}^{n-1} p_{ni}^{(n)}} = \sum_{i=1}^{n-1} \pi_i \frac{p_{in}^{(n)}}{S(n)} \).
If $\pi_j = kr_j$ with $r_1 = 1$ then

$$\sum_{i=1}^{N} \pi_i = 1 \Rightarrow k = 1/\sum_{i=1}^{N} r_i \text{ with}$$

$$r_n = \frac{\sum_{i=1}^{n-1} r_i p(n)}{S(n)}, (n = 2, \ldots, N),$$

implying

$$\pi_i = \frac{r_i}{\sum_{n=1}^{N} r_n}, \quad i = 1, 2, \ldots, N.$$
\[
\begin{array}{cccccccc}
\rho^{(2)}_{11} & \rho^{(2)}_{12} & \rho^{(2)}_{13} & \rho^{(n)}_{1N-2} & \rho^{(n)}_{1N-1} & \rho^{(n)}_{1N} \\
\rho^{(2)}_{21} & \rho^{(3)}_{22} & \rho^{(3)}_{23} & \rho^{(n)}_{2n-2} & \rho^{(n)}_{2n-1} & \rho^{(n)}_{2n} \\
\rho^{(3)}_{31} & \rho^{(3)}_{32} & \rho^{(n)}_{3n-2} & \rho^{(n)}_{3n-1} & \rho^{(n)}_{3n} \\
\rho^{n-1}_{n-1,1} & \rho^{n-1}_{n-1,2} & \rho^{n-1}_{n-1,3} & \rho^{n-1}_{n-1,n-2} & \rho^{n-1}_{n-1,n-1} \\
\rho^{n}_{n,1} & \rho^{n}_{n,2} & \rho^{n}_{n,3} & \rho^{n}_{n,n-2} & \rho^{n}_{n,n-1} \\
\rho^{N-1}_{N-1,1} & \rho^{N-1}_{N-1,2} & \rho^{N-1}_{N-1,3} & \rho^{N-1}_{N-1,n-2} & \rho^{N-1}_{N-1,n-1} \\
\rho^{N}_{N,1} & \rho^{N}_{N,2} & \rho^{N}_{N,3} & \rho^{N}_{N,n-2} & \rho^{N}_{N,n-1} \\
\rho^{N-2}_{N-2,1} & \rho^{N-2}_{N-2,2} & \rho^{N-2}_{N-2,3} & \rho^{N-2}_{N-2,n-2} & \rho^{N-2}_{N-2,n-1} & \rho^{N-2}_{N-2,n} \\
\rho^{N-1}_{N-1,1} & \rho^{N-1}_{N-1,2} & \rho^{N-1}_{N-1,3} & \rho^{N-1}_{N-1,n-2} & \rho^{N-1}_{N-1,n-1} & \rho^{N-1}_{N-1,n} \\
\rho^{N}_{N,1} & \rho^{N}_{N,2} & \rho^{N}_{N,3} & \rho^{N}_{N,n-2} & \rho^{N}_{N,n-1} & \rho^{N}_{N,n} \\
\end{array}
\]
GTH Algorithm

1. Start with a Markov chain with $N$ states and transition matrix $P_N = \begin{bmatrix} p_{ij}^{(N)} \end{bmatrix}$.

2. Compute for $n = N, N-1, \ldots, 3$,

$$p_{ij}^{(n-1)} = p_{ij}^{(n)} + \frac{p_{in}^{(n)} p_{nj}^{(n)}}{S(n)}, 1 \leq i \leq n-1, 1 \leq j \leq n-1; \text{ where}$$

$$S(n) = \sum_{j=1}^{n-1} p_{nj}^{(n)}.$$

3. Set $r_1 = 1$ and compute $r_n = \frac{\sum_{i=1}^{n-1} r_i p_{in}^{(n)}}{S(n)}$, for $n = 2, \ldots, N$.

4. Compute $\pi_i = \frac{r_i}{\sum_{j=1}^{N} r_j}$, $i = 1, 2, \ldots, N$. 
7. Mean First Passage Times via Extended GTH

We seek a computational procedure, utilising the GTH/State reduction procedure.

For a M.C. \( \{X_n\} \) with \( N \)-states and transition matrix \( P \), its mean first passage time matrix (MFPT) \( M \) satisfies

\[
(I - P)M = E - PM_d
\]

where \( E = [1] = e^{(N)}e^{(N)T} \) and

\[
M_d = \begin{bmatrix} \delta_{ij}m_{jj} \end{bmatrix} = \text{diag}(\pi_1, \pi_2, \ldots, \pi_N).
\]

For a M.R.P. \( \{X_n, T_n\} \) the MFPT matrix satisfies

\[
(I - P)M = \mu^{(N)}e^{(N)T} - P(M)_d.
\]
**M.R.P Primer**
From Hunter (1982)

Let \( \{(X_n, T_n)\}, \ (n \geq 0) \), be a Markov renewal process (M.R.P.) with state space \( S_N \) and semi-Markov kernel \( Q(t) = \begin{bmatrix} Q_{ij}(t) \end{bmatrix} \), where 

\[
Q_{ij}(t) = P\{X_{n+1} = j, \ T_{n+1} - T_n \leq t \mid X_n = i\}, \ (i, j) \in S_N.
\]

\( X_n \) is the state at the \( n \)-th transition
\( T_n \) is the time of the \( n \)-th transition.

Let 
\[
P = \begin{bmatrix} p_{ij} \end{bmatrix}
\]
be the transition matrix of the embedded M.C. \( \{ X_n \}, \ (n \geq 0) \), \( p_{ij} = Q_{ij}(+\infty) = P\{X_{n+1} = j \mid X_n = i\} \).

\[
Q_{ij}(t) = p_{ij} F_{ij}(t) \quad \text{where} \quad F_{ij}(t) = P\{T_{n+1} - T_n \leq t \mid X_n = i, \ X_{n+1} = j\}.
\]
$F_{ij}(t)$ is the distribution function of the “holding time” $T_{n+1} - T_n$ in state $X_n$ until transition into state $X_{n+1}$ given that the M.R.P. makes a transition from $X_n$ to $X_{n+1}$.

Let $\mu_{ij} = \int_0^\infty t \, dQ_{ij}(t)$ so that $\mu_{ij} = p_{ij} E[T_{n+1} - T_n \mid X_n = i, X_{n+1} = j]$.

Let $P^{(1)} = \begin{bmatrix} \mu_{ij} \end{bmatrix}$ then

$$(I - P)M = P^{(1)}E - PM_d.$$

Let $\mu = P^{(1)}e$ then $\mu^T = (\mu_1, \mu_2, \ldots, \mu_N)$ where $\mu_i = \sum_{j=1}^N \mu_{ij}$. $\mu_i = E[T_{n+1} - T_n \mid X_n = i]$ is the “mean holding time in state $i$ “. Thus $P^{(1)}E = P^{(1)}ee^T = \mu e^T$.

Note that for a M.C. $\mu^{(N)T} = e^{(N)T} = (1,1,\ldots,1)$ and $P^{(1)}E = E$. 
Let us partition $M = M_N$ as $M_N = \begin{bmatrix} M_{N-1} & m_{N-1}^{(N)(c)} \\ m_{N-1}^{(N)(r)T} & m_{NN} \end{bmatrix}$

where

$M_{N-1} = \begin{bmatrix} m_{ij} \end{bmatrix}$, $(1 \leq i \leq N - 1, 1 \leq j \leq N - 1)$,

$m_{N-1}^{(N)(r)T} = (m_{N1}, m_{N2}, \ldots, m_{N,N-1})$ and

$m_{N-1}^{(N)(c)T} = (m_{1N}, m_{2N}, \ldots, m_{N-1,N})$.

Let us also partition $\mu^{(N)T} = (\mu_1^{(N)}, \ldots, \mu_{N-1}^{(N)}, \mu_N^{(N)}) = (\mu_{N-1}^{(N)T}, \mu_N^{(N)})$

where $\mu_{N-1}^{(N)T} = (\mu_1^{(N)}, \ldots, \mu_{N-1}^{(N)})$

Expressing $(I - P)M = \mu^{(N)T} e^{(N)T} - P(M)_{d}$ in block form and carrying out block multiplication we obtain the following results (details omitted).
Using the expression for $P_{N-1}$, as derived for the GTH algorithm, it is easily seen that

$$(I_{N-1} - P_{N-1})M_{N-1} = \mu^{(N-1)} e^{(N-1)^T} - P_{N-1}(M_{N-1})_d,$$

where

$$\mu^{(N-1)} = \mu^{(N)} + \frac{\mu^{(N)} p^{(N)(c)}}{p^{(N)(r)} e^{(N-1)}}.$$

Further, 

$$m_{N-1}^{(N)(r)T} = \left\{ p^{(N)(r)T} M_{N-1} - (M_{N-1})_d \right\} + \mu^{(N)} e^{(N-1)^T}$$

implying

$$m_{Nj} = \left\{ \sum_{k=1, k \neq j}^{N-1} p^{(N)} m_{kj} + \mu^{(N)} \right\} \frac{S(N)}{S(N)}$$

for $1 \leq j \leq N - 1$,

leading to expressions for $m_{Nj}$ in terms of $m_{1j}, \ldots, m_{kj}, \ldots, m_{N-1,j}$ $(k \neq j)$, i.e. expressions for $m_{Nj}$ in terms of the remaining elements of the $j$ – th column of $M$. 
More difficult to find $m_{N-1}^{(N)(c)}$, i.e. the $m_{iN}$ for $1 \leq i \leq N - 1$. 

$$(l_{n-1} - Q_{n-1}^{(n)})m_{n-1}^{(n)(c)} = \mu_{n-1}^{(n)}$$

$Q_{N-1}^{(N)} = \begin{bmatrix} p_{ij}^{(N)} \end{bmatrix}$ for $1 \leq i \leq N - 1$, $1 \leq j \leq N - 1$, an $(n-1)\times(n-1)$ matrix derived from $P_N$, requires further step by step reduction procedure by eliminating $m_{N-1,N}$ from $m_{N-1}^{(N)(c)T}$ replacing it in the expressions for the elements $m_{1N}, m_{2N}, \ldots, m_{N-2,N}$.

Need to express $(N-1)\times(N-1)$ matrix $Q_{N-1}^{(N)}$ in block form.

$$m_{N-1,N} = \frac{\left\{ p_{N-2}^{(N-1)(N)(r)} T m_{N-2}^{(N)(c)} + \mu_{N-1}^{(N)} \right\}}{1 - p_{N-1,N-1}^{(N-1)}} = \left\{ \sum_{k=1}^{N-2} q_{N-1,k}^{(N-1)} m_{kN} + \mu_{N-1}^{(N)} \right\}$$

where $R(N) = 1 - p_{N-1,N-1}^{(N-1)} = \sum_{j=1, j \neq N-1}^{N} p_{N-1,j}^{(N)}$ (i.e. obtained from $P_N$).
Thus for a general reduction from $n$ states to $n-1$ states

If $(I_n - P_n)M_n = \mu^{(n)}e^{(n)T} - P_n(M_n)_d$ where $\mu^{(n)T} = (\mu_{n-1}^{(n)}, \mu_n^{(n)})$,

then $(I_{n-1} - P_{n-1})M_{n-1} = \mu^{(n-1)}e^{(n-1)T} - P_{n-1}(M_{n-1})_d$

where $\mu^{(n-1)T} = \mu_{n-1}^{(n)T} + \frac{\mu_n^{(n)}p_{n-1}^{(n)(c)T}}{p_{n-1}^{(n)(r)T}e^{(n-1)}}$.

$\mu^{(n)T} = (\mu_{n-1}^{(n)T}, \mu_n^{(n)})$ is a $1 \times n$ vector, $\mu_{n-1}^{(n)T} = (\mu_1^{(n)},...,\mu_{n-1}^{(n)})$ and

$\mu^{(n-1)T} = (\mu_1^{(n-1)},...,\mu_{n-1}^{(n-1)})$ is a $1 \times (n-1)$ vector, with

$$\mu_i^{(n-1)} = \mu_i^{(n)} + \frac{\mu_n^{(n)}p_{i,n}^{(n)}}{S(n)}, \quad (1 \leq i \leq n-1).$$

where $S(n) = p_{n-1}^{(n)(r)T}e^{(n-1)} = \sum_{j=1}^{n-1}p_{nj}^{(n)} = 1 - p_{nn}^{(n)}$. 
We can reduce the state space by 1 at successive steps retaining the same mean first passage times for the reduced state space i.e. $M_{n-1} = \begin{bmatrix} m_{ij} \end{bmatrix}$, for $1 \leq i \leq n-1$, $1 \leq j \leq n-1$, although the calculation is modified with mean holding times in the states being modified. i.e. in effect we are using a MRP variant to preserve the mean first passage times for the reduced state space.

If we are given $M_{n-1} = \begin{bmatrix} m_{ij} \end{bmatrix}$, $(1 \leq i \leq n-1$, $1 \leq j \leq n-1)$, we wish to find $m_{n-1}^{(n)(c)}$, $m_{n-1}^{(n)(r)T}$ and $m_{nn}$.

First $m_{nn} = 1/\pi_n^{(N)}$ so we can use the GTH algorithm from the calculation of the stationary probabilities.

For $m_{n-1}^{(n)(c)}$, $m_{nj} = \left\{ \begin{array}{l} \sum_{k=1, k \neq j}^{n-1} p_{nk}^{(n)} m_{kj} + \mu_{(n)}^n \\ S(n) \end{array} \right\}$ for $1 \leq j \leq n-1$. 
For $n = 2$:

\[(I_2 - P_2)M_2 = \mu^{(2)e^{(2)T}} - P_2(M_2)\]

\[
\begin{bmatrix}
1 - p_{11}^{(2)} & -p_{12}^{(2)} \\
-p_{21}^{(2)} & 1 - p_{22}^{(2)}
\end{bmatrix}
\begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\mu_1^{(2)} & \mu_1^{(2)} \\
\mu_2^{(2)} & \mu_2^{(2)}
\end{bmatrix}
- \begin{bmatrix}
p_{11}^{(2)}m_{11} & p_{12}^{(2)}m_{22} \\
p_{21}^{(2)}m_{11} & p_{22}^{(2)}m_{22}
\end{bmatrix}
\]

leading to

\[
M_2 = \begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix}
= \begin{bmatrix}
\frac{p_{21}^{(2)}\mu_1^{(2)} + p_{12}^{(2)}\mu_2^{(2)}}{p_{21}^{(2)}} & \frac{\mu_1^{(2)}}{p_{12}^{(2)}} \\
\frac{\mu_2^{(2)}}{p_{21}^{(2)}} & \frac{p_{21}^{(2)}\mu_1^{(2)} + p_{12}^{(2)}\mu_2^{(2)}}{p_{12}^{(2)}}
\end{bmatrix}
\]
General procedure for finding all the elements of $M$.

Step 1: Start with $P_N$ and concentrate on finding only the expressions for $m_{i1}$ for $i = 1, 2, \ldots, N$.
i.e. if $P_N = \begin{bmatrix} p_{ij}^{(N)} \end{bmatrix}$ carry out the extended GTH algorithm
For $n = N, N - 1, \ldots, 3$,
let $p_{ij}^{(n-1)} = p_{ij}^{(n)} + \frac{p_{in}^{(n)} p_{nj}^{(n)}}{S(n)}$, $1 \leq i \leq n - 1, 1 \leq j \leq n - 1$
and $\mu_i^{(n-1)} = \mu_i^{(n)} + \frac{\mu_i^{(n)} p_{i,n}^{(n)}}{S(n)}$, $(1 \leq i \leq n - 1)$, with $S(n) = \sum_{j=1}^{n-1} p_{nj}^{(n)}$.
with $(\mu_1^{(N)}, \mu_2^{(N)}, \ldots, \mu_N^{(N)}) = (1, 1, \ldots, 1)$. 
Let $m_{11} = \mu_1^{(2)} + \frac{p_{12}^{(2)} \mu_2^{(2)}}{p_{21}^{(2)}}$,

$$m_{21} = \frac{\mu_2^{(2)}}{S(2)},$$

$$m_{31} = \frac{p_{32}^{(3)} m_{21} + \mu_3^{(3)}}{S(3)},$$

$$m_{n1} = \sum_{k=2}^{n-1} \frac{p_{nk}^{(n)} m_{k1} + \mu_n^{(n)}}{S(n)}, \quad n = 3, \ldots, N.$$

This provides the entries of the first column of $M = \begin{bmatrix} m_{ij} \end{bmatrix}$, i.e. $m_{N}^{(1)(N)}$, where

$$M = (m_{N}^{(1)(N)}, m_{N}^{(2)(N)}, \ldots, m_{N}^{(N)(N)}) \text{ with } m_{N}^{(1)(N)T} = (m_{11}, m_{21}, \ldots, m_{N1})$$
Step 2: Now reorder the rows of $P^{(N)}$ by moving the first column after the $N$th column, followed by moving the first row to the last row.

$$P_N \equiv P_N^{(1)} = \begin{bmatrix}
    p_{11} & p_{12} & \ldots & p_{1,N-1} & p_{1,N} \\
    p_{21} & p_{22} & \ldots & p_{2,N-1} & p_{2,N} \\
    p_{N-1.1} & p_{N-1,2} & \ldots & p_{N-1,N-1} & p_{N-1,N} \\
    p_{N1} & p_{N2} & \ldots & p_{N,N-1} & p_{NN} \\
    \end{bmatrix} \rightarrow \begin{bmatrix}
    p_{22} & p_{2,N-1} & p_{2N} & p_{21} \\
    p_{N-1,2} & p_{N-1,N-1} & p_{N-1,N} & p_{N-1,1} \\
    p_{N2} & p_{N,N-1} & p_{NN} & p_{N,1} \\
    p_{12} & p_{1,N-1} & p_{1,N} & p_{11} \\
    \end{bmatrix} \equiv P_N^{(2)}$$
Step 3: Carry out the algorithm, as in Step 1, with \( P_N = P_N^{(2)} \)
to obtain the vector of MFPTs which we label as \( \mathbf{m}_N^{(2)(N)} \)
where \( \mathbf{m}_N^{(2)(N)T} = (m_{22}, m_{32}, \ldots, m_{N2}, m_{12}) \).

Step 4: Reorder \( P_2^{(N)} \) as in step 2 to obtain \( P_3^{(N)} \) and repeat Step 3
to obtain \( \mathbf{m}^{(3)(N)} \) where \( \mathbf{m}^{(3)(N)T} = (m_{33}, m_{43}, \ldots, m_{N3}, m_{13}, m_{23}) \).

Step \( k \): Repeat as above with \( P_k^{(N)} \) to obtain \( \mathbf{m}^{(k)(N)} \) where
\[
\mathbf{m}^{(k)(N)T} = (m_{kk}, m_{k+1,k}, \ldots, m_{N,k}, m_{1,k}, \ldots, m_{k-1,k})
\]
finishing with \( P_N^{(N)} \) and \( \mathbf{m}^{(N)(N)} \) where \( \mathbf{m}^{(N)(N)T} = (m_{NN}, m_{1,N}, m_{2,N}, \ldots, m_{N-1,N}) \).

Step \( N + 1 \): Let \( \overline{M} = (\mathbf{m}_N^{(1)(N)} , \mathbf{m}_N^{(2)(N)} , \ldots , \mathbf{m}_N^{(N)(N)}) \).
Finally reorder \( \overline{M} \) to obtain \( M = (\mathbf{m}_N^{(1)(N)} , \mathbf{m}_N^{(2)(N)} , \ldots , \mathbf{m}_N^{(N)(N)}) \).
This can be carried out using the following MatLab procedure:

```matlab
for col=1:m
    for row= 1:m
        P_new1(mod(row+m-2,m)+1,col)=P(row,col);
    end
end
for col=1:m
    for row= 1:m
        P_new2(row,mod(col+m-2,m)+1)=P_new1(row,col);
    end
end
P=P_new2;
PP=P;
end
for col=1:m
    for row=1:m
        M1(mod(row+col-2,m)+1,col)=M(row,col);
    end
```

This can be carried out using the following MatLab procedure:
8. Test Problems

Introduced by Harrod & Plemmons (1984) and considered by others in different contexts.

**TP1**: The original transition matrix was not irreducible and was replaced (Heyman (1987), Heyman & Reeves (1989)) by

\[
\begin{bmatrix}
0.1 & 0.6 & 0 & 0.3 & 0 & 0 \\
0.5 & 0.5 & 0 & 0 & 0 & 0 \\
0.5 & 0.2 & 0 & 0 & 0.3 & 0 \\
0 & 0.7 & 0 & 0.2 & 0 & 0.1 \\
0.1 & 0 & 0.8 & 0 & 0 & 0.1 \\
0.4 & 0 & 0.4 & 0 & 0 & 0.2 \\
\end{bmatrix}
\]
TP2 (Also Benzi (2004))

\[
\begin{bmatrix}
.85 & 0 & .149 & .0009 & 0 & .00005 & 0 & .00005 \\
.1 & .65 & .249 & 0 & .00009 & .00005 & 0 & .00005 \\
.1 & .8 & .99996 & .0003 & 0 & 0 & .0001 & 0 \\
0 & .0004 & 0 & .7 & .2995 & 0 & .0001 & 0 \\
.0005 & 0 & .0004 & .399 & .6 & .0001 & 0 & 0 \\
0 & .00005 & 0 & 0 & .00005 & .6 & .2499 & .15 \\
.00003 & 0 & .00003 & .00004 & 0 & .1 & .8 & .0999 \\
0 & .00005 & 0 & 0 & .00005 & .1999 & .25 & .55
\end{bmatrix}
\]
<table>
<thead>
<tr>
<th></th>
<th>0.999999</th>
<th>(1.0 \times 10^{-7})</th>
<th>(2.0 \times 10^{-7})</th>
<th>(3.0 \times 10^{-7})</th>
<th>(4.0 \times 10^{-7})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>(5.0 \times 10^{-7})</td>
<td>0</td>
<td>0.999999</td>
<td>0</td>
<td>5.0 (\times 10^{-7})</td>
<td></td>
</tr>
<tr>
<td>(5.0 \times 10^{-7})</td>
<td>0</td>
<td>0</td>
<td>0.999999</td>
<td>5.0 (\times 10^{-7})</td>
<td></td>
</tr>
<tr>
<td>(2.0 \times 10^{-7})</td>
<td>(3.0 \times 10^{-7})</td>
<td>(1.0 \times 10^{-7})</td>
<td>(4.0 \times 10^{-7})</td>
<td>0.999999</td>
<td></td>
</tr>
</tbody>
</table>
**TP4 variants:**  \(TP41 \equiv \varepsilon = 1.0\times10^{-1}, TP42 \equiv \varepsilon = 1.0\times10^{-3},\)

\(TP43 \equiv \varepsilon = 1.0\times10^{-5}, TP44 \equiv \varepsilon = 1.0\times10^{-7}.\)

\[
\begin{bmatrix}
.1-\varepsilon & .3 & .1 & .2 & .3 & \varepsilon & 0 & 0 & 0 & 0 \\
.2 & .1 & .1 & .2 & .4 & 0 & 0 & 0 & 0 & 0 \\
.1 & .2 & .2 & .4 & .1 & 0 & 0 & 0 & 0 & 0 \\
.4 & .2 & .1 & .2 & .1 & 0 & 0 & 0 & 0 & 0 \\
.6 & .3 & 0 & 0 & 0 & .1 & 0 & 0 & 0 & 0 \\
\varepsilon & 0 & 0 & 0 & 0 & .1-\varepsilon & .2 & .2 & .4 & .1 \\
0 & 0 & 0 & 0 & 0 & .2 & .2 & .1 & .3 & .2 \\
0 & 0 & 0 & 0 & 0 & .1 & .5 & 0 & .2 & .2 \\
0 & 0 & 0 & 0 & 0 & .5 & .2 & .1 & 0 & .2 \\
0 & 0 & 0 & 0 & 0 & .1 & .2 & .2 & .3 & .2 \\
\end{bmatrix}
\]
9. Comparisons

We present comparisons for the test problems, the 4 algorithms (Standard, Simple, Perturbations and Extended GTH), under double precision, for the MFPT matrix $M$ and compute the MAX RESIDUAL ERRORS:

\[
\text{MAX RES ERROR} = \max_{1 \leq i \leq m, 1 \leq j \leq m} |m_{ij} - \sum_{k \neq j} p_{ik} m_{kj} - 1|
\]
<table>
<thead>
<tr>
<th>Max Res Error</th>
<th>M_Standard</th>
<th>M_Simple</th>
<th>M_Pert</th>
<th>M_EGTH</th>
</tr>
</thead>
<tbody>
<tr>
<td>TP1</td>
<td>5.6843e-14</td>
<td>5.6843e-14</td>
<td>1.1369e-13</td>
<td>1.1369e-13</td>
</tr>
<tr>
<td>TP2</td>
<td>1.8190e-12</td>
<td>1.8190-12</td>
<td>3.6380e-12</td>
<td>3.6380e-12</td>
</tr>
<tr>
<td>TP3</td>
<td>1.7027</td>
<td>1.7594</td>
<td>1.5073</td>
<td>1.6188</td>
</tr>
<tr>
<td>TP41</td>
<td>1.4211e-14</td>
<td>2.1316e-14</td>
<td>1.4211e-14</td>
<td>1.4211e-14</td>
</tr>
<tr>
<td>TP42</td>
<td>1.0374e-12</td>
<td>1.8190e-12</td>
<td>9.0950e-13</td>
<td>1.8190e-12</td>
</tr>
<tr>
<td>TP43</td>
<td>1.1642e-10</td>
<td>1.7462e-10</td>
<td>1.7462e-10</td>
<td>1.1642e-10</td>
</tr>
<tr>
<td>TP44</td>
<td>1.4901e-08</td>
<td>1.4901e-08</td>
<td>1.4901e-08</td>
<td>7.4506e-09</td>
</tr>
</tbody>
</table>