Deciding the Isomorphism Problem in Classes of Unary Automatic Structures

Jiamou Liu\textsuperscript{a}, Mia Minnes\textsuperscript{b}

\textsuperscript{a}Department of Computer Science, University of Auckland, Auckland, New Zealand
\textsuperscript{b}Department of Mathematics, MIT, Cambridge, MA, USA

Abstract

We solve the isomorphism problem for certain classes of unary automatic structures: unary automatic equivalence relations, unary automatic linear orders, and unary automatic trees. That is, we provide algorithms which decide whether two given elements of these classes are isomorphic. In doing so, we define new finite representations for these structures which give normal forms. \textsuperscript{1}

Key words: Unary automatic structures, the isomorphism problem, graphs

1. Introduction

The isomorphism problem, which asks for a decision procedure to decide whether two given members of a class of structures are isomorphic, is central in studying the effective content of mathematical objects. Over finite structures, the isomorphism problem has been one of the most challenging problems in complexity theory (it belongs to NP but is neither known to be in P nor known to be NP-complete) [1]. Over computable structures (those where the domain and atomic relations of the structure are computable), it is well-known that the isomorphism problem is undecidable; in fact, it is complete for $\Sigma^0_1$, the first level of the analytical hierarchy [2]. In [3], Khoussainov and Nerode initiate a systematic study of automatic structures, those where elements are encoded as strings over a finite alphabet and whose domain and atomic relations are represented by finite automata (precise definitions in Section 2). Automatic structures form an intermediate class of structures between the finite structures and the computable structures. This paper focuses on the isomorphism problem for unary automatic structures, the subclass of automatic structures (which still contains all finite structures) of structures whose domains are encoded as strings over a one letter alphabet.

\textsuperscript{1}A related paper which focuses on time and space complexity for these unary automatic structures has been submitted for publication by the authors elsewhere.
Automatic structures have decidable first-order theories [3]. In general, their monadic second-order theories (where quantification over sets is permitted) are undecidable, see [4, 5, 6] for an overview of automatic structures. Since key applications for automatic structures include modeling databases [7] and verifying programs [8], applying the transitive closure operator is often desirable. However, this operator is expressible in monadic second-order logic but is not first-order definable: reachability is undecidable for automatic structures in general. On the other hand, unary automatic structures have decidable monadic second-order theories.

The restriction to a unary alphabet is a natural special case of automatic structures because any automatic structure has an isomorphic copy over the binary alphabet [5]. Moreover, if we consider the intermediate class of structures whose domain elements are encoded as finite strings over $1^*2^*$, insufficient decidability strength results: since the infinite grid can be coded automatically over $1^*2^*$ and counter machines can be coded into the grid, reachability is not decidable in this class of structures. Thus, the class of unary automatic structures is a sensible context where reachability is decidable.

The broad decidability of unary automatic structures can be exploited when they are used to model streaming databases. Databases with entries encoded as strings of 1s are well-suited to situations in which provisional results must be updated on the fly (using the tally representation) and computations are performed in real time.

In this paper, we study the isomorphism problem in classes of unary automatic structures. These structures include unary automatic equivalence relations, linear orders, and trees. We use (known and new) characterizations of members of these classes to get normal forms and polynomial-time (in these normal forms) algorithms for the isomorphism problem.

The isomorphism problem has been studied for other collections of graphs. For automatic graphs, it is $\Sigma_1^1$-complete [9]. On the other hand, the isomorphism problem is decidable for equational graphs [10]. Any monadic second-order expressible question is decidable in the class of unary automatic graphs. However, the isomorphism problem is not a priori expressible in this way and it is not known whether it is decidable. This paper works towards a solution of this question by looking at special subclasses of unary automatic graphs.

Many natural graph problems (such as graph connectivity and reachability) are expressible in monadic second-order logic and are hence decidable for unary automatic graphs. Deciding these questions by a translation of monadic second-order formulæ yields very slow algorithms (non-elementary complexity). Khoussainov, Liu and Minnes [11] exploited structural properties of unary automatic graphs with finite degree to solve these questions in polynomial-time.

In general, understanding the structural properties of a class of unary automatic structures leads to more efficient algorithms. Khoussainov and Rubin [12] and Blumensath [4] characterized unary automatic graphs in terms of relations between two finite graphs. This led to characterizations of unary automatic linear orders and equivalence structures as well (see Theorems 4.2 and 5.2). We prove an analogous result for unary automatic trees (see Theorem 6.3). We use these structural characterizations to define concise finite representations of unary automatic structures. These representations lead to polynomial-time algorithms for the isomorphism problem.
Paper Organization. Section 2 recalls the definitions of finite automata and automatic structures. Section 3 discusses the special case of unary automatic structures. Sections 4, 5, and 6 discuss equivalence structures, linear orders, and trees (respectively). We give polynomial-time algorithms solving the isomorphism problem for the class of structures considered in each section. We conclude in Section 7 and mention open questions.

The authors would like to thank the anonymous referees for helpful comments and corrections.

2. Preliminaries

A finite automaton is a restricted Turing machine which has a fixed finite bound on its resources and is allowed only a single read pass over the input data. More formally, a finite automaton \( A \) over a finite alphabet \( \Sigma \) is a tuple \( (S, q_0, \Delta, F) \), where \( S \) is a finite set of states, \( q_0 \in S \) is the initial state, \( \Delta : S \times \Sigma \rightarrow S \) is the transition function, and \( F \subset S \) is the set of final or accepting states. In this paper we require \( \Delta \) to be a well-defined total function and hence \( A \) is a complete and deterministic automaton. An input to \( A \) is a finite string in \( \Sigma^* \); the empty string is denoted by \( \lambda \). A computation of \( A \) on a word \( \sigma_1\sigma_2\ldots \sigma_n \in \Sigma^* \) is a sequence of states, say \( q_0, q_1, \ldots, q_n \), such that \( q_0 = i \) and \( (q_i, \sigma_{i+1}, q_{i+1}) \in \Delta \) for all \( i \in \{0, 1, \ldots, n - 1\} \). If \( q_n \in F \) then the computation is successful and the automaton \( A \) accepts the word. As such, if \( q \in F \) we say that \( q \) is an accepting state. The language of \( A \), \( L(A) \), is the set of all words accepted by \( A \). In general, \( D \subset \Sigma^* \) is FA recognizable, or regular, if \( D \) is the language of some finite automaton. If \( A \) is a finite automaton over the unary alphabet \{1\} it is called a unary automaton and its language is a unary automatic subset of \{1\}∗.

A (relational) structure \( S \) consists of a countable domain \( D \) and atomic relations on \( D \). We focus on structures with a single binary relation \( S = (D; R) \). Synchronous 2-tape automata recognize binary relations. Such automata have two input tapes, each of which contains one of the input words. Bits of the two input words are read in parallel at the same rate until both input strings have been completely processed. Formally, let \( \Sigma_0 = \Sigma \cup \{\diamond\} \) where \( \diamond \) is a symbol not in \( \Sigma \). Given a pair of words \( w_1, w_2 \in \Sigma^* \), the convolution of \( (w_1, w_2) \) is a word \( \otimes(w_1, w_2) \) over the alphabet \( (\Sigma_0)^2 \) with length \( \max(|w_1|, |w_2|) \). The \( k^{th} \) symbol of \( \otimes(w_1, w_2) \) is \( (\sigma_1, \sigma_2) \) where \( \sigma_i \) is the \( k^{th} \) symbol of \( w_i \) if \( k \leq |w_i| \), and is \( \diamond \) otherwise. A binary relation \( R \) is FA recognizable if the set of convolutions of all pairs \( (w_1, w_2) \in R \) is a regular subset of \( (\Sigma_0^2)^* \).

A structure is called automatic over \( \Sigma \) if its domain is a regular subset of \( \Sigma^* \) and each of its basic relations is FA recognizable. A structure is called unary automatic if it is automatic over the alphabet \{1\}. The structures \((\mathbb{N}; S)\) and \((\mathbb{N}; \leq)\) are both isomorphic to unary automatic structures. On the other hand, \((\mathbb{Q}; \leq)\) and \((\mathbb{N}; +)\) have isomorphic copies which are automatic over \( \{0, 1\} \) but have no unary automatic isomorphic copies. The structure \((\mathbb{N}; \times)\) has no automatic isomorphic copy. For proofs of these facts, see the survey papers [13, 14].

The class of languages recognizable by finite automata is closed under the rational operations studied by Kleene. These operations parallel Boolean set operations and the first-order quantifiers. Consider the first-order logic extended by \( \exists^* \) (there exist
Table 1: Deciding properties of binary relations in automatic structures.

<table>
<thead>
<tr>
<th>Property</th>
<th>First-order definition</th>
<th>Time complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflexivity</td>
<td>∀x (R(x, x))</td>
<td>O(mn)</td>
</tr>
<tr>
<td>Symmetry</td>
<td>∀x, y (R(x, y) → R(y, x))</td>
<td>O(n^2)</td>
</tr>
<tr>
<td>Antisymmetry</td>
<td>∀x, y (R(x, y) ∧ R(y, x) → x = y)</td>
<td>O(n^3)</td>
</tr>
<tr>
<td>Totality</td>
<td>∀x, y (R(x, y) ∨ R(y, x))</td>
<td>O(mn^2)</td>
</tr>
<tr>
<td>Transitivity</td>
<td>∀x, y, z (R(x, y) ∨ R(y, z) → R(x, z))</td>
<td>O(n^3)</td>
</tr>
</tbody>
</table>

Theorem 2.1 (Blumensath, Grädel; 1999. Khoussainov, Rubin; 1999). For an automatic structure \( S \) there is an algorithm that, given a formula \( \varphi(\vec{x}) \) in \( \text{FO} + \exists^\omega + \exists^m \), produces an automaton whose language is those tuples \( \vec{a} \) from \( S \) that make \( \varphi \) true.

The proof of Theorem 2.1 along with the fact that there is a linear-time algorithm which tests whether the language of a given automaton is empty yield the following corollaries.

Corollary 2.2. The \( (\text{FO} + \exists^\omega + \exists^m) \)-theory of an automatic structure \( S \) is decidable.

Corollary 2.3. Given deterministic automata \( \mathcal{A}_1 \) (m states) and \( \mathcal{A}_2 \) (n states), there is an \( O(mn) \)-time algorithm to build the deterministic union or intersection automaton (mn states) of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) and an \( O(m) \)-time algorithm to build the deterministic complement automaton (m states) of \( \mathcal{A}_1 \).

Let \( (D, R) \) be an automatic structure (over \( \Sigma \)), with \( R \) a binary relation over \( D \). Suppose \( \mathcal{A}_D \) (m states) and \( \mathcal{A}_R \) (n states) are deterministic finite automata recognizing \( D \) and \( R \), respectively. Some first-order definable properties of binary relations are listed in Table 1. By Corollary 2.2, it is decidable whether \( (D, R) \) satisfies these properties. In particular, to check if \( R \) is reflexive, we construct an automaton for \( \{x \mid (x, x) \in R\} \) and check if \( \{x \mid (x, x) \in R\} \cap D = D \). Similarly, to decide if \( R \) is symmetric, we construct an automaton \( \mathcal{A}_1 \) recognizing the relation \( \{(y, x) \mid (x, y) \in R\} \) and check if \( R = L(\mathcal{A}_1) \). For antisymmetry, we construct an automaton for \( S = \{(x, y) \mid x \neq y\} \) and determine whether \( R \cap R_1 \cap S = \emptyset \). To decide if \( R \) is total, it suffices to check whether \( R \cup L(\mathcal{A}_1) = D^2 \). Finally, to settle whether \( R \) is transitive, we construct the automaton \( \{(x, y, z) \mid R(x, y) \land R(y, z) \land \neg R(x, z)\} \) and ask whether its language is empty. Note that if \( D = \Sigma^* \) then \( m = 1 \).

3. Unary Automatic Structures

This section explores unary automatic structures and introduces terminology and notation used throughout the paper. Recall that a structure is \textit{unary automatic} if it is
automatic over the alphabet \{1\}. We use \( x \) to denote the string \( 1^* \) and \( \mathbb{N} \) for the set of all such strings \( \{1\}^* \). The following lemma from [4] characterizes the regular subsets of \( \{1\}^* \).

**Lemma 3.1 (Blumensath; 1999).** A set \( L \subseteq \mathbb{N} \) is regular over the alphabet \( \{1\}^* \) if and only if there are numbers \( t, \ell \in \mathbb{N} \) such that \( L = L_1 \cup L_2 \) with \( L_1 \subseteq \{ x : x < t \} \) and \( L_2 \) the finite union \( \bigcup_{j=0,1,...,r-1} \{ t + j \ell : k \in \mathbb{N} \} \) where \( k_j < \ell \) for all \( j \).

**Proof.** We describe the shape of an arbitrary deterministic 1-tape unary automaton \( A = (S, \iota, \Delta, F) \). If \( n = |S| \) there are \( t, \ell \leq n \) so that the following holds. There is a sequence of states \( S_1 = \{ q_1, q_2, \ldots, q_t \} \) such that \( \Delta(\iota, 1) = q_1 \) and for all \( 1 \leq i < t \), \( \Delta(q_i, 1) = q_{i+1} \). There is another sequence of states \( S_2 = \{ q_{t+1}, \ldots, q_{t+r} \} \) such that for all \( t \leq j < l \), \( \Delta(q_j, 1) = q_{j+1} \), and \( \Delta(q_l, 1) = q_{t+1} \). Every final state in \( S_1 \) recognizes exactly one word less than \( t \), and every final state in \( S_2 \) recognizes the set of all words \( t + i \ell + k, i \in \omega \), for some fixed \( k < \ell \). The language of such an automaton has the form described in the statement of the lemma; given an \( L \) from the statement of the lemma and its parameters \( t, \ell \), we can define the corresponding unary automaton. \( \square \)

Synchronous 2-tape unary automata recognize binary relations over \( \mathbb{N} \). The general shape of these automata is given in Figure 1. We fix some terminology. States reachable from the initial state by reading inputs of type \( (1, 1) \) are called \( (1, 1) \)-states. The set of \( (1, 1) \)-states is a disjoint union of a tail and a loop. We label the \( (1, 1) \)-states as \( q_0, \ldots, q_t, \ldots, q_m \) where \( q_0, \ldots, q_{t-1} \) form the \( (1, 1) \)-tail and there is a transition from \( q_m \) to \( q_t \) to close the \( (1, 1) \)-loop. States reachable from a \( (1, 1) \)-state by reading inputs of type \( (1, \diamond) \) are called \( (1, \diamond) \)-states. The set of \( (1, \diamond) \)-states reachable from any given \( q_i \) consist of a tail and a loop, called the \( (1, \diamond) \)-tail and \( (1, \diamond) \)-loop from \( q_i \), respectively. The \( (\diamond, 1) \)-tails and loops are defined similarly.

![Figure 1: General shape of a deterministic 2-tape unary automaton.](image)

Khoussainov and Rubin [12] and Blumensath [4] generalized Lemma 3.1 and gave a characterization of all binary relations on \( \mathbb{N} \) which are recognized by some synchronous 2-tape automaton. In particular, if we view such a relation as the edge relation on the graph of nodes labelled by \( \mathbb{N} \), the characterization relates all the unary automatic graphs to an unwinding or ladder of finite graphs. Let \( \mathcal{B} = (B, E_B) \) and
\[ \mathcal{D} = (D, E_D) \] be finite graphs. Let \( R_1, R_2 \) be subsets of \( D \times B \), and let \( R_3, R_4 \) be subsets of \( B \times B \). Consider the graph \( \mathcal{D} \) followed by countably infinitely many copies of \( \mathcal{B} \), ordered as \( \mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2, \ldots \). We define the infinite graph \( \text{unwind}(\mathcal{B}, \mathcal{D}, \mathcal{R}) \) as follows. Its vertex set is \( D \cup B^0 \cup B^1 \cup B^2 \cup \ldots \) and its edge set contains \( E_D \cup E^0 \cup E^1 \cup \ldots \) as well as the following edges, for all \( a, b \in B \), \( d \in D \), and \( i, j \in \omega \):

- \((d, b^0)\) when \((d, b) \in R_1\), and \((d, b^{i+1})\) when \((d, b) \in R_2\),
- \((a', b^{i+1})\) when \((a, b) \in R_3\), and \((a', b^{i+2+j})\) when \((a, b) \in R_4\).

**Lemma 3.2** (Blumensath; 1999. Khoussainov, Rubin; 2001). A graph is unary automatic if and only if it is isomorphic to \( \text{unwind}(\mathcal{B}, \mathcal{D}, \mathcal{R}) \) for some finite graphs \( \mathcal{B}, \mathcal{D} \) and relations on these graphs given by \( \mathcal{R} \).

Proof. Suppose a graph is unary automatic and its edge relation is recognized by a synchronous 2-tape automaton \( \mathcal{A} \). Using the terminology from above, we define the vertices of \( \mathcal{D} \) to be the states on the \((1, 1)\)-tail of \( \mathcal{A} \). The edges of \( \mathcal{D} \) are determined by (some of) the accepting states on the \((\cdot, 1)\)- and \((1, \cdot)\)-tails off the \((1, 1)\)-tail. Similarly, the vertices of \( \mathcal{B} \) are the states on the \((1, 1)\)-loop of \( \mathcal{A} \). The \( R_i \) relations are determined by the appropriate accepting states on the \((1, \cdot)\)- and \((\cdot, 1)\)-tails off the \((1, 1)\) states of \( \mathcal{A} \). Reversing this construction gives a synchronous 2-tape automaton recognizing the edge relation of a graph isomorphic to a given unwinding. In Figure 2, we provide an example of an automaton and unwinding pair to clarify the construction. □

![Figure 2: An example of unwind(\mathcal{B}, \mathcal{D}, \mathcal{R}) and the synchronous 2-tape automaton for its edge relation. If we label \mathcal{B} = \{a, b\} and \mathcal{D} = \{0, 1, 2\} then \( E_D = \{(0, 1)\}, E_B = \emptyset, R_1 = \{(1, a), (2, b)\}, R_2 = \{(2, b)\}, R_3 = \{(a, a)\}, \) and \( R_4 = \emptyset \).](image)

In this paper we restrict our attention to (countably) infinite structures. The following lemma allows us to assume that the domain of each structure is \( \mathbb{N} \) (rather than a regular subset of \( \mathbb{N} \)) without increasing the size of the associated unary automaton.

**Lemma 3.3.** Let \( (D; R) \) be a unary automatic structure with \( D \subset \mathbb{N} \). Suppose this structure is presented by \( \mathcal{A}_D \) and \( \mathcal{A}_R \). There is a deterministic 2-tape unary automaton \( \mathcal{A}_R' \), \( |\mathcal{A}_R'| \leq |\mathcal{A}_R| \), such that \( (\mathbb{N}; L(\mathcal{A}_R')) \cong (D; R) \).

Proof. Let \( t \) and \( \ell \) be as described in Lemma 3.1. We outline the proof in the case when the parameter \( t \) associated with \( D \) is 0. Since \( R \) is a binary relation over the domain \( D \), \( \mathcal{A}_R \) must satisfy the following requirements: the \((1, 1)\)-tail has length \( c' \ell \) for some constant \( c' \); the \((1, 1)\)-loop has length \( c \ell \) for some constant \( c \); the lengths of all loops
and tails containing accepting states are multiples of \(\ell\); and, there are no accepting states on any tail or loops off any \((1, 1)\)-states of the form \(q_{cwh}\) where \(h \neq k_j\) (where \(k_j\) is as defined in Lemma 3.1). The isomorphism between \(D\) and \(N\) will be given by \(i\ell + k_j \mapsto i\ell + j\). Therefore, define \(\mathcal{AR}_R\) to have a \((1, 1)\)-tail of length \(c' r\), a \((1, 1)\)-loop of length \(c r\), and copy the information from the state \(i\ell + j\) in \(\mathcal{AR}_R\) to state \(i\ell + j\) in \(\mathcal{AR}'_R\) (modifying the lengths of \((\circ, 1)\)- and \((1, \circ)\)-tails and loops appropriately). Then, \((N; L(\mathcal{AR}_R)) \equiv (D; R)\) and since \(r \leq \ell\), \(\mathcal{AR}'_R\) has no more states than \(\mathcal{AR}_R\). \(\square\)

Algorithms on unary automatic binary relations have as input a deterministic synchronous 2-tape unary automaton recognizing the relation. The size of the input is defined to be the number of states in this automaton. We say that a synchronous 2-tape automaton is standard if the lengths of all its loops and tails equal some number \(p\), called the loop constant. If \(\mathcal{A}\) is a standard automaton with \(n\) states and loop constant \(p\), then \(n = 8p^2\).

Lemma 3.4. For each deterministic 2-tape unary automaton with \(n\) states there is an equivalent standard automaton with at most \(8n^2\) states.

Proof. Let \(p\) be the least common multiple of the lengths of all loops and tails of \(\mathcal{A}\). An easy estimate shows that \(p\) is no more than \(n^4\). One can transform \(\mathcal{A}\) into an equivalent standard automaton whose loop constant is \(p\). Hence, there is a standard automaton equivalent to \(\mathcal{A}\) whose size is bounded above by \(8n^2\). \(\square\)

By Lemma 3.4, we assume all unary automatic structures are presented using standard automata. This assumption in general incurs a super-exponential cost in the state space. However, the standard automata provide natural normal forms for the structures and allow smoother algorithms.

We fix some notation for a standard automaton \(\mathcal{A}\) with loop constant \(p\). The \((1, 1)\)-states are labelled \(q_0, \ldots, q_{2p-1}\) as described above. For \(0 \leq j < 2p\), let \(W_j = \{x \in \mathbb{N} : \Delta(q_0, (1, 1)^x) = q_j\}\). Then \(W_0, \ldots, W_{2p-1}\) partition \(\mathbb{N}\) and we have \(W_j = \{j\}\) for \(0 \leq j < p\), \(W_j = \{j + ip : i \in \mathbb{N}\}\) for \(p \leq j < 2p\). We enumerate the elements of \(W_j\) as \(v'_j = j + ip\).

4. Unary Automatic Equivalence Relations

This section explores unary automatic equivalence relations. A structure \(\mathcal{E} = (\mathbb{N}; E)\) is an equivalence structure if \(E\) is an equivalence relation (reflexive, symmetric, and transitive). By Table 1, there is an \(O(n^3)\) time algorithm that decides whether a given synchronous 2-tape unary automaton presents an equivalence relation. The main theorem of this section is the following.

Theorem 4.1. The isomorphism problem for unary automatic equivalence structures is decidable in linear time in the sizes of the input standard automata.

The height, \(h^0\), of an equivalence structure \(\mathcal{E}\) is a function \(\mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\}\) such that \(h^0(x)\) is the number of \(E\)-equivalence classes of size \(x\). Two equivalence structures \(\mathcal{E}_1\) and \(\mathcal{E}_2\) are isomorphic if and only if \(h^0_{\mathcal{E}_1} = h^0_{\mathcal{E}_2}\). By the following characterization
from [4, 12], heights of unary automatic equivalence structures are finitely nonzero
and $h_E^0(\infty) \neq \infty$. If $k$ is the size of the largest finite equivalence class of $E$, $h_E^0$ can be
encoded by the finite function $h_E$ with domain $k + 2$ such that $h_E(i) = h_E^0(i)$ for $i \leq k$
and $h_E(k + 1) = h_E^0(\infty)$.

**Theorem 4.2** (Blumensath; 1999. Khoussainov, Rubin; 2001). An equivalence struc-
ture has a unary automatic presentation if and only if it has finitely many infinite equi-
valence classes and there is a finite bound on the sizes of the finite equivalence classes.

Let $E$ be recognized by a standard automaton $A = (S, \delta, \Delta, F)$ with loop constant $p$
(and hence $n = |S| = 8p^2$). Recall the definitions of $q_j$ and $W_j$ from Section 3. Observe
that since equivalence relations are symmetric, for any $0 \leq j < 2p$, a state on the $(1, \diamond)$-
tail or loop of $q_j$ is accepting if and only if the corresponding state on the $(\diamond, 1)$-tail or
loop is also accepting.

**Lemma 4.3.** For $0 \leq j < 2p$, each element of $W_j$ belongs to an infinite equivalence
class if and only if the $(1, \diamond)$-loop from $q_j$ has an accepting state. Moreover, in this
case, $W_j$ forms a subset of some equivalence class.

**Proof.** If the $(1, \diamond)$-loop from $q_j$ contains an accepting state, then for each $x \in W_j$,
there exists infinitely many $y$ with $(x, y) \in E$. Suppose further that $j \geq p$ (if $j < p$ then
$W_j = \{j\}$ so we’re done). Then, $j$ is equivalent to $j + p(i + 1) + D$ for some $0 \leq D < p$
and all $i > 0$. But, since $q_j$ is on the $(1, 1)$-loop, the accepting state on the $(1, \diamond)$-loop
of $q_j$ also gives that $j + p$ is equivalent to $j + p(i + 2) + D$ for the same $D$ and all $i > 0$.
Transitivity and symmetry then imply that $j$, $j + p$ are equivalent, hence all members of
$W_j$ are equivalent. On the other hand, suppose the $(1, \diamond)$-loop from $q_j$ does not contain
any accepting states. Then, for each $i$, the equivalence class of $j + pi$ must be a subset
of $0, \ldots, j + p(i + 1) - 1$, a finite set. \hfill \Box

**Lemma 4.4.** For $0 \leq j < 2p$, if $W_j$ does not belong to an infinite equivalence class
then it is in an equivalence class of size less than $p$.

**Proof.** Suppose $q_j$ has no accepting state on its $(1, \diamond)$-loop. Define $j_0$ to be the least
number in the (finite) equivalence class containing $j$. The size of the equivalence class
each $x \in W_j$ is the number of accepting states on the $(1, \diamond)$-tail from $q_{j_0}$. \hfill \Box

**Lemma 4.5.** Given $A$, Algorithm 1 computes the graph of $h_E$ in time $O(n)$.

**Proof.** By Lemma 4.3, the size of finite equivalence classes is bounded by the size
of $(1, \diamond)$-tails, hence $h_E^0(n) = 0$ for any $n > p$. By Lemma 4.4, for all $x$, if $h_E^0(x)$ is
finite then $h_E^0(x) \leq p$. Algorithm 1 exploits the transitivity of the equivalence relation
to reduce the number of states of the automaton which must be visited. Moreover,
for each $j$ that is considered, the algorithm must check whether at most $4p$ states are
accepting. Thus, the runtime of Algorithm 1 is $O(p^2) = O(n)$. \hfill \Box

**Proof (of Theorem 4.1).** Let $\mathcal{A}_1$ ($n$ states) and $\mathcal{A}_2$ ($m$ states) be standard automata
recognizing equivalence relations $E_1, E_2 \subseteq \mathbb{N}^2$. By Lemma 4.5, extracting $h_{E_1}$ and $h_{E_2}$
takes time $O(\max(m, n))$. By Lemma 4.3, $\text{dom}(h_{E_1}) \cup \text{dom}(h_{E_2}) \subseteq \max(m + 1, n + 1)$.
Therefore, checking if $h_{E_1} = h_{E_2}$ takes $O(\max(m, n))$. \hfill \Box
5 UNARY AUTOMATIC LINEAR ORDERS

Algorithm 1 Equivalence Relation Height
1: Initialize array \( h[0 \ldots p+1] \) to 0. Create list \( L = 0, \ldots, 2p-1 \).
2: while \( L \neq \emptyset \) do
3: \( j = \) least in \( L \).
4: if the \((1, \odot)\)-loop from \( q_j \) contains no accepting states then
5: \( j < p \) then
6: Add 1 to \( h(\)number of accepting states on \((1, \odot)\)-tail from \( q_j \))\).
7: else
8: Set \( h(\)number of accepting states on \((1, \odot)\)-tail from \( q_j \)) to \( p+1 \).
9: end if
10: Remove the indices of all these accepting states from \( L \).
11: else
12: Increment \( h(p+1) \) by 1.
13: Remove the indices of all accepting states on \((1, \odot)\)-tail and loop from \( L \).
14: end if
15: end while

5. Unary Automatic Linear Orders

This section studies unary automatic linear orders. A linear order is total, reflexive, anti-symmetric, and transitive. By Table 1, checking if a binary relation recognized by an \( n \)-state unary automaton is a linear order takes \( O(n^3) \). We prove the following theorem.

Theorem 5.1. The isomorphism problem for unary automatic linear orders is decidable in linear time in the sizes of the input standard automata.

The following theorem from [4, 12] describes which linear orders have unary automatic presentations. We use \( \omega \) to denote the order type of the natural numbers, \( \omega^* \) to denote the order type of the negative integers, and \( 1 \) to denote the singleton linear order.

Theorem 5.2 (Blumensath; 1999. Khoussainov, Rubin; 2001). A linear order \( \mathcal{L} = (L; \leq_L) \) is unary automatic if and only if it is isomorphic to a finite sum of linear orders of type \( \omega, \omega^* \) or \( 1 \).

By Theorem 5.2, each unary automatic linear order \( \mathcal{L} \) can be written as a finite word \( u_0u_1 \ldots u_k \) over the alphabet \( \{1, \omega, \omega^*\}^2 \). The canonical word, \( w_\mathcal{L} \), of \( \mathcal{L} \) is the minimal such word; \( 1 \omega \) and \( \omega^* 1 \) do not appear as substrings of \( w_\mathcal{L} \). Two unary automatic linear orders \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are isomorphic if and only if \( w_{\mathcal{L}_1} = w_{\mathcal{L}_2} \). Let \( \mathcal{L} = (\mathbb{N}; \leq_L) \) be a linear order recognized by a standard unary automaton \( \mathcal{A} \) with loop constant \( p \). Recall the definitions of \( q_j \) and \( W_j \) from Section 3.

The following lemmas describe the possible relationships between \( W_j \) and \( W_k \) for \( j < k \). It will be convenient to assign names to all states on the \((\odot, 1)\)-tails and loops;

\[ \text{This word denotes the linear order } u_0 + u_1 + \cdots + u_k \]
these names will be suggestive of the respective relationships. Note that since the linear
order is total, whether states on the \((1, \odot)\)-tails and loops are accepting or rejecting is
completely determined by the \((\odot, 1)\)-tails and loops.

For \(0 \leq j < k < p\),
\[
q_{j<k} := \Delta(q_j, (\odot, 1)^{k-j}).
\]

For \(0 \leq j < p\) and \(p \leq k < j\),
\[
q_{j<k} := \Delta(q_j, (\odot, 1)^{k-j}),
\]
\[
q_{j<k} := \Delta(q_j, (\odot, 1)^{p+k-j}).
\]

For \(0 \leq j < p\) and \(p + j \leq k < 2p\),
\[
q_{j<k} := \Delta(q_j, (\odot, 1)^{k-j}),
\]
\[
q_{j<k} := \Delta(q_j, (\odot, 1)^{p+k-j}).
\]

For \(p \leq j < k < 2p\),
\[
q_{j<k} := \Delta(q_j, (\odot, 1)^{k-j}),
\]
\[
q_{j<k} := \Delta(q_j, (\odot, 1)^{p+k-j}).
\]

For \(p \leq j < 2p\),
\[
q_i := \Delta(q_j, (\odot, 1)^p).
\]

**Lemma 5.3.** For \(p \leq j < 2p\), \(W_j\) either forms an infinite increasing chain or an infinite
decreasing chain.

Proof. If \(q_{i'} \in F\) then for all \(i \in \mathbb{N}\), \(v_i^j <_L v_{i+1}^j\). Thus, the sequence \(v_0^j, v_1^j, v_2^j, \ldots\) is
an infinite increasing chain. If not, then \(q_{i'} \notin F\) implies that \(\Delta(q_j, (1, \odot)^{i'}) \in F\). Thus,
\(v_0^j, v_1^j, v_2^j, \ldots\) is an infinite decreasing chain. \(\Box\)

Hence, there is an \(O(n)\) test checking if a given \(W_j\) is an increasing chain or a
decreasing chain.

**Lemma 5.4.** For \(p \leq j < 2p\), \(W_j\) is a subset of one copy of \(\omega\) or one copy of \(\omega^*\) in \(L\).

Proof. By Lemma 5.3, it is sufficient to prove that any two elements in \(W_j\) are sepa-
rated by at most finitely many elements of \(L\). Consider \(v_i^j, v_j^i\) with \(i < i'\). Suppose
\[
v_i^j <_L 2j + p(i + i' + r) <_L v_i^j
\]
for some \(r \geq 1\) and \(s \geq 0\). By the first inequality, \(\Delta(q_j, (\odot, 1)^{i+i'+s+j}) \in F\). By the
second inequality, \(\Delta(q_j, (1, \odot)^{i+i'+s+j}) \in F\). This contradicts the anti-symmetry of \(L\).

Therefore, any \(z\) such that \(v_i^j <_L z <_L v_j^i\) must satisfy \(z < 2j + (i + i' + 1)p\); there are
only finitely many such \(z\). \(\Box\)

**Lemma 5.5.** Let \(p \leq j < k < 2p\). If \(q_{j<k} \in F \land q_{k<j} \notin F\) then \(W_j\) precedes \(W_k\):
\[
\forall x \in W_j \forall y \in W_k (x <_L y).
\]

Similarly, if \(q_{j<k} \notin F \land q_{k<j} \in F\) then \(W_k\) precedes \(W_j\):
\[
\forall x \in W_j \forall y \in W_k (y <_L x).
\]
Proof. Suppose \( q'_{jck} \in F \) and \( q'_{k<} \notin F \). We first show it must be the case that \( q'_{jck} \in F \) and \( q'_{k<} \notin F \). Assume for a contradiction that \( q'_{jck} \notin F \). Then (for any \( i \))

\[
v^i_{t+2} \preceq_w v^k_i \preceq_w v^i_{t+1} \preceq_w v^k_{t+2}
\]

but also, \( v^k_{t+2} \preceq_w v^j_i \), contradicting antisymmetry. Similarly, assume for a contradiction that \( q'_{k<} \in F \). Then (for any \( i \))

\[
v^i_{t+3} \preceq_w v^k_i \preceq_w v^j_{t+1} \preceq_w v^k_{t+2}
\]

and \( v^k_{t+2} \preceq_w v^j_{t+3} \), a contradiction. Thus \( v_i^j <_L v_i^k \) for any \( i, r \). Hence, \( W_j \) precedes \( W_k \).

An analogous argument shows that if we assume that \( q'_{jck} \notin F \land q'_{k<} \notin F \) then \( q'_{jck} \in F \) and \( q'_{k<} \in F \). Thus, in this case, \( W_k \) precedes \( W_j \).

\[\Box\]

**Lemma 5.6.** Let \( p \leq j < k < 2p \). If \( q'_{jck} \in F \land q'_{k<} \in F \) then \( W_j \) and \( W_k \) interleave within the same copy of \( \omega \) in \( L \). If \( q'_{jck} \notin F \land q'_{k<} \notin F \) then \( W_j \) and \( W_k \) interleave within the same copy of \( \omega^* \) in \( L \).

Proof. Suppose \( q'_{jck} \in F \) and \( q'_{k<} \in F \). Then for all \( i \), \( v^i_1 <_L v^j_{t+2} <_L v^k_{t+3} <_L v^j_{t+5} \). In particular, this implies \( W_j \) and \( W_k \) are both increasing. Moreover, there are constants \( C, d \in \mathbb{Z} \) (depending on which of \( q'_{jck} \) and \( q'_{k<} \) are final) such that \( v^i_j <_L v^i_{red} <_L v^i_{red+1} \) for all \( i \geq C \). Using Lemma 5.4, we conclude that \( W_j \) and \( W_k \) are in the same copy of \( \omega \) in \( L \). Symmetrically, if \( q'_{jck} \notin F \) and \( q'_{k<} \notin F \) then \( W_j \) and \( W_k \) are both decreasing and they are in the same copy of \( \omega^* \) in \( L \).

\[\Box\]

The proof in Lemma 5.6 can be slightly generalized to see that for \( p \leq h < j < k < 2p \), if \( W_h, W_j \) interleave and \( W_j, W_k \) interleave then \( W_h, W_k \) interleave.

**Lemma 5.7.** For \( 0 \leq j < p \) and \( p \leq k < 2p \), \( \{j\} \) interleave with \( W_k \) if and only if \( p \leq k < p + j \) and

\[
q'_{jck} \in F \iff q'_{jck} \notin F.
\]

Proof. It is only possible for \( \{j\} \) to interleave with \( W_k \) if it is ordered in a different way with respect to \( v^k_0 \) than with respect to \( v^i_k \) for \( i > 0 \). If \( p + j < k < 2p \) then all elements of \( W_k \) are represented on the \((o,1)\)-loop off \( q_j \) and so the ordering of \( j \) with respect to all of them is determined by \( q'_{jck} \). So, we suppose \( p \leq k < p + j \). If \( q'_{jck} \in F \iff q'_{jck} \notin F \), then either \( j <_L v^i_j \) for all \( i \) or \( v^i_j <_L f \) for all \( i \). Thus, in this case, there is no interleave. Finally, consider the case where \( q'_{jck} \in F \) but \( q'_{jck} \notin F \). Then, for all \( i > 0 \),

\[
v^i_k <_L j <_L v^0_i.
\]

By Lemma 5.3, this implies that \( W_k \) is part of an \( \omega^* \) chain, and that \( j \) is interleaved in this chain. The symmetric case \( (q'_{jck} \notin F \text{ but } q'_{jck} \in F) \) is analogous.

\[\Box\]

**Lemma 5.8.** Algorithm 2 extracts \( w_L \) from \( \mathcal{A} \) in time \( O(n) \).
Proof. Informally, the algorithm works from least to greatest elements in $L$, checking whether relevant states in $A$ are accepting or rejecting to build up $w_L$. More precisely, we define and use sets Left($i$) to indicate which $W_j$ precede $W_i$. We notice that Lemma 5.6 allows us to partition $\{p, \ldots, 2p - 1\}$ into sets each of which correspond to a single copy of $\omega$ or $\omega^*$ in $w_L$. Using Lemma 5.7, we add to these sets some elements in $\{0, \ldots, p - 1\}$ which fall inside these chains. In Algorithm 2, the resulting sets are labelled $V$. The computation of the sets Left($i$), $V$ requires visiting each $(\diamond, 1)$-state at most once. Once these sets have been computed the algorithm must check whether at most $p$ many states are in $F$. Thus, the algorithm runs in $O(n + p^2) = O(n)$. □

Proof (of Theorem 5.1). Given two standard automata $A_1$ (with $n$ states) and $A_2$ (with $m$ states) recognizing linear orders $\leq_{L_1}, \leq_{L_2} \subseteq \mathbb{N}^2$, Lemma 5.8 gives $w_{L_1}$ and $w_{L_2}$ in time $O(\max\{m, n\})$. □

6. Unary Automatic Trees

We now turn to unary automatic trees. A structure $T = (T; \leq_T)$ is a tree if $\leq_T$ is a partial order on $T$ (reflexive, antisymmetric, and transitive) with a root (least element) and such that for all nodes $x \in T$, the set $\{y : y \leq_T x\}$ is a finite linear order. Table 1 lists efficient tests for most of the requirements for being a tree. However, checking if $\leq_T$ has a root requires verifying the first-order sentence $\exists x \forall y (x \leq_T y)$. The alternation of quantifiers implies an exponential-time decision procedure for general automatic binary relations [17]. This can be improved for unary automatic binary relations.

Lemma 6.1. If $(\mathbb{N}; R)$ is a partial order where $R$ is recognized by a unary automaton (not necessarily standard) with $n$ states, there is an $O(n)$ time algorithm which checks for an $R$-least element.

Proof. Let $m$ be the number of $(1, 1)$-states in $A$. If there is an $R$-least element $x$, then $x < m$. Indeed, if $x \geq m$, there is $y < m$ such that $\Delta(x, (1, 1)^y) = \Delta(y, (1, 1)^y)$. Let $q = \Delta(x, (1, 1)^x)$. Because $x$ is an $R$-least element we have that $R(x, y)$ and so $\Delta(q, (1, \diamond)^{x-y}) \in F$; similarly, $R(x, 2x - y)$ implies that $\Delta(q, (\diamond, 1)^{x-y}) \in F$. However, this means that $R(y, x)$, a contradiction with anti-symmetry of $R$.

The $R$-least element $x$ (if it exists), must satisfy that for all $z < x < y$

$$\Delta(q_z, (\diamond, 1)^{y-x}) \in F \quad \text{and} \quad \Delta(q_z, (1, \diamond)^{x-y}) \in F.$$ 

Reading each $(\diamond, 1)$ and $(1, \diamond)$ state at most once is sufficient to find such an $R$-least element or decide that one doesn’t exist. (Note that we are using our assumption from Section 1 that the given unary automaton is complete.) In particular, Algorithm 3 does this and runs in $O(n)$.

Combining Lemma 6.1 with Table 1 gives the following theorem.

Theorem 6.2. There exists an $O(n^4)$ time algorithm that decides if a unary automatic binary structure is a tree.
Algorithm 2 Linear order canonical word

1: Initialize $B = \emptyset$, $w = \lambda$. Create list $L = 0, \ldots, 2p - 1$.
2: Initialize each Left$(i) = \emptyset$. Initialize an array of sets $V_{\ell}$ all to be empty.
3: for each $1$-state given by pair $j < k$ do
4: \hspace{1em} if $0 < j < k < p$ then
5: \hspace{2em} if $q_{j<k} \in F$, put $j \in \text{Left}(k)$;
6: \hspace{2em} if $q_{j<k} \notin F$, put $k \in \text{Left}(j)$.
7: \hspace{1em} else if $0 \leq j < p$, $p \leq k < p + j$ then
8: \hspace{2em} if $q_{j<k} \in F$, $q_{j<k} \notin F$, put $j \in \text{Left}(k)$;
9: \hspace{2em} if $q_{j<k} \notin F$, $q_{j<k} \notin F$, put $k \in \text{Left}(j)$;
10: \hspace{2em} if $q_{j<k} \in F$, $q_{j<k} \notin F$, remove $j$ from $L$ and ensure $(j, k) \subset V_{\ell}$ for some $\ell$;
11: \hspace{2em} if $q_{j<k} \notin F$, $q_{j<k} \in F$, remove $j$ from $L$ and ensure $(j, k) \subset V_{\ell}$ for some $\ell$.
12: \hspace{1em} else if $0 \leq j < p$, $p + j \leq k < 2p$ then
13: \hspace{2em} if $q_{j<k} \in F$, put $j \in \text{Left}(k)$;
14: \hspace{2em} if $q_{j<k} \notin F$, put $k \in \text{Left}(j)$.
15: \hspace{1em} else if $p \leq j < k < 2p$ then
16: \hspace{2em} if $q_{j<k} \in F$, $q_{j<k} \notin F$, put $j \in \text{Left}(k)$;
17: \hspace{2em} if $q_{j<k} \notin F$, $q_{j<k} \in F$, put $k \in \text{Left}(j)$;
18: \hspace{2em} if $q_{j<k} \in F$, $q_{j<k} \in F$, remove $k$ from $L$ and ensure $(j, k) \subset V_{\ell}$ for some $\ell$;
19: \hspace{2em} if $q_{j<k} \notin F$, $q_{j<k} \notin F$, remove $k$ from $L$ and ensure $(j, k) \subset V_{\ell}$ for some $\ell$.
20: \hspace{1em} end if
21: \hspace{1em} end for
22: while $L \neq \emptyset$ do
23: \hspace{1em} Let $i$ be least in $L$ such that Right$(i) = B$
24: \hspace{1em} if $i < p$ then
25: \hspace{2em} Put $B = B \cup \{i\}$ and $w = w \cdot 1$. Remove $i$ from $L$.
26: \hspace{1em} else
27: \hspace{2em} Let $j$ be least such that $i, j \in V_{\ell}$ for some $\ell$.
28: \hspace{2em} if $W_j$ forms an increasing chain then
29: \hspace{3em} Put $B = B \cup V_{\ell}$ and $w = w \cdot \omega$. For $k \in V_{\ell}$, remove $k$ from $L$.
30: \hspace{2em} else
31: \hspace{3em} Put $B = B \cup V_{\ell}$ and $w = w \cdot \omega^*$. For $k \in V_{\ell}$, remove $k$ from $L$.
32: \hspace{2em} end if
33: \hspace{1em} end if
34: \hspace{1em} end while
35: In $w$, combine any $1 \cdot \omega$ to $\omega$ and any $\omega^* \cdot 1$ to $\omega^*$. 
Algorithm 3 \( R \)-least element

1: Initialize the list \( L = 0, \ldots, m - 1 \).
2: \textbf{while} \( L \neq \emptyset \) \textbf{do}
3: \hspace{1em} Let \( j \) be the first element in \( L \).
4: \hspace{1em} \textbf{if} all \((\emptyset, 1)\)-states out of \( q_j \) are accepting \textbf{then}
5: \hspace{1em} \hspace{1em} \( j \) is the \( R \)-least element; return \text{true}.
6: \hspace{1em} \textbf{else}
7: \hspace{1em} \hspace{1em} delete \( j \) from \( L \).
8: \hspace{1em} \hspace{1em} \textbf{for} \( k \in L \) \textbf{do}
9: \hspace{1em} \hspace{1em} \hspace{1em} \textbf{if} \( \Delta(q_j, (\emptyset, 1)^{k-j}) \in F \) \textbf{then} delete \( k \) from \( L \).
10: \hspace{1em} \hspace{1em} \hspace{1em} \textbf{if} \( \Delta(q_j, (1, \emptyset)^{k-j}) \notin F \) \textbf{then} delete \( k \) from \( L \).
11: \hspace{1em} \hspace{1em} \textbf{end for}
12: \hspace{1em} \textbf{end if}
13: \hspace{1em} \textbf{end while}
14: \textbf{return} \text{false}

Theorem 6.3 is a characterization of unary automatic trees which will lead to an efficient algorithm for the isomorphism problem. This theorem is similar in spirit to the unwinding description of unary automatic graphs in [12, 4] that was discussed as Lemma 3.2. A parameter set \( \Gamma \) is a tuple \( (T_0, T_1, \ldots, T_m, \sigma, X) \) where \( T_0, T_1, \ldots, T_m \) are finite trees, \( \sigma : \{1, \ldots, m\} \to T_0 \), and \( X : \{1, \ldots, m\} \to \emptyset \cup \bigcup_i T_i \) such that \( X(i) \in T_i \cup \{\emptyset\} \). A tree-unfolding of a parameter set \( \Gamma \) is a tree \( \text{UF}(\Gamma) \) that contains one copy of \( T_0 \) and infinitely many copies of \( T_i \) for each \( i \in \{1, \ldots, m\} \) connected as follows. The root of \( \text{UF}(\Gamma) \) is the root of \( T_0 \). For \( i \in \{1, \ldots, m\} \), if \( X(i) \neq \emptyset \), the root of the first copy of \( T_i \) is an immediate descendent of \( \sigma(i) \) and the root of each subsequent copy of \( T_i \) is an immediate descendent of the copy of \( X(i) \) in the previous copy of \( T_i \). Otherwise, if \( X(i) = \emptyset \), the root of each copy of \( T_i \) is an immediate descendent of \( \sigma(i) \).

Theorem 6.3. A tree \( \mathcal{T} = (\mathbb{N}, \leq_T) \) is unary automatic if and only if there is a parameter set \( \Gamma = (T_0, T_1, \ldots, T_m, \sigma, X) \) such that \( \mathcal{T} \cong \text{UF}(\Gamma) \).

We will need a few definitions and lemmas to prove this theorem. Suppose \( \leq_T \) is recognized by a standard automaton \( \mathcal{A} \) with loop constant \( p \). Recall from Section 5 the definition of \( W_j \) and the labels of states of \( \mathcal{A} \). In particular, we will use the notations \( q_{j<k}^l \) and \( q_{j<k}^l \). However, since \( \leq_T \) is a partial (rather than linear) order, the \((1, \emptyset)\) states are not determined by their \((\emptyset, 1)\) counterparts. Hence, we use \( q_{j<k}^l \) and \( q_{j<k}^l \) to denote the appropriate \((1, \emptyset)\) states. Two nodes \( x, y \in \mathcal{T} \) are incomparable, \( x \not\leq_T y \), if \( x \not\leq_T y \) and \( y \not\leq_T x \). For \( p \leq j < 2p \), \( W_j \) is a chain if \( v_i^j <_T v_i^j <_T \ldots \); \( W_j \) is an anti-chain if \( v_i^j \not<_T v_k^j \) for all \( i \neq k \).

Lemma 6.4. For \( p \leq j < 2p \), \( W_j \) is a chain or an anti-chain. Also, \( W_j \) is a chain if and only if for each \( x \in W_j \), the set \( \{ y : x <_T y \} \) is infinite.

Proof. Let \( p \leq j < 2p \). Suppose \( q_{j<p}^l \in F \). Then \( v_i^j <_T v_{i+1}^j <_T v_{i+2}^j \) for all \( i \). Hence, \( W_j \) is a chain and for any \( x \in W_j \), the set \( \{ y : x <_T y \} \) is infinite.
On the other hand, suppose \( q^i_j \notin F \). Since \( T \) is a tree, there are no infinite \( \prec_T \)-descending chains. Hence, \( \Delta(q_j, (1, o)^p) \notin F \). Therefore, for any \( r, v^i_j \notin_T v^i_i, \) and \( v^i_{i+1} \notin_T v^i_i \) and \( W_j \) is an anti-chain. Assume for a contradiction that there is some \( i \) such that \( \{ v : v^i_j \prec_T y \} \) is infinite. In particular, there is some \( k \) such that \( \{ v^k_j : v^i_j \prec_T v^k_j \} \) is infinite and so \( q^i_{j,k} \in F \). Hence, \( v^i_j, v^i_{i+1} \prec_T v^i_{i+2} \). Since the set of \( \prec_T \) predecessors of \( v^i_{i+2} \) is linearly ordered, \( v^i_j \prec_T v^i_{i+1} \) or \( v^i_j \prec_T v^i_i \), a contradiction.

Prima facie, there are \( 2^{10} \) many possibilities for the interactions between \( W_j \) and \( W_k \) in the tree order since each interaction is determined by whether each of the following states is accepting or not:

\[
q^i_j, q^o_j, q^i_{j,k}, q^o_{j,k}, q^i_{j,k}, q^o_{j,k}; q^i_{k,j}, q^o_{k,j}; q^i_{k,j}, q^o_{k,j}.
\]

However, we can use the fact that \( \mathcal{A} \) recognizes a tree partial order to eliminate the possibilities dramatically. The following lemma collects the requisite observations; it is proved using properties of trees, such as that the set of predecessors of any tree element is finite and linearly ordered.

**Lemma 6.5.** Let \( p \leq j < k < 2p \).

1. \( q^i_{j,k} \notin F \land q^o_{k,j} \notin F \).
2. \( \neg(q^i_{j,k} \in F \land q^o_{j,k} \in F) \). \( \neg(q^o_{k,j} \in F \land q^i_{k,j} \in F) \).
3. \( \neg(q^i_{j,k} \in F \land q^o_{k,j} \in F) \). \( \neg(q^o_{j,k} \in F \land q^i_{j,k} \in F) \).
4. If \( q^i_{j,k} \notin F \), then \( q^o_{j,k} \notin F \). If \( q^o_{j,k} \notin F \), then \( q^i_{j,k} \notin F \).
5. If \( q^i_{j,k} \in F \), then \( \neg(q^i_{j,k} \notin F \land q^o_{k,j} \notin F) \). If \( q^o_{j,k} \in F \), then \( \neg(q^o_{j,k} \notin F \land q^i_{k,j} \notin F) \).
6. If \( q^o_{j,k} \in F \), then \( \neg(q^o_{j,k} \notin F \land q^i_{j,k} \notin F) \). If \( q^i_{j,k} \in F \), then \( \neg(q^i_{j,k} \notin F \land q^o_{k,j} \notin F) \).
7. If \( q^i_{j,k} \in F \) and \( q^o_{j,k} \notin F \), then \( q^i_{k,j} \notin F \land q^o_{j,k} \notin F \).
   If \( q^i_{j,k} \notin F \) and \( q^o_{j,k} \in F \), then \( q^o_{j,k} \notin F \land q^i_{k,j} \notin F \).

Figure 3: An example of a tree-unfolding.
Lemma 6.7. If \( q^{i}_{j < k} \in F \) and \( q^{i}_{k} \in F \), then \( \neg (q^{i}_{j < k} \in F \land q^{i}_{k} \notin F) \).

9. If \( q^{i}_{j} \in F \) and \( q^{i}_{k} \in F \), then

\[
d^{i}_{j < k} \not\in F \land d^{i}_{j} \not\in F \implies \left[ q^{i}_{j < k} \notin F \land q^{i}_{k} \notin F \land d^{i}_{j} \not\in F \right].
\]

Lemma 6.5 allows us to conclude that if both \( W_{j} \) and \( W_{k} \) are antichains then \( d^{i}_{j < k} \not\in F \) and \( d^{i}_{k} \not\in F \) and at most one of \( d^{i}_{j < k}, d^{i}_{j}, d^{i}_{k} \) is in \( F \). If both \( W_{j} \) and \( W_{k} \) are increasing chains then Lemma 6.5 shows that whether \( d^{i}_{j < k}, d^{i}_{j}, d^{i}_{k} \) are accepting or rejecting completely determines the values of the other variables. In the case where \( W_{j} \) is an increasing chain but \( W_{k} \) is an antichain, we see that \( d^{i}_{j < k}, d^{i}_{j}, d^{i}_{k} \) can not be in \( F \). Moreover, the value of \( d^{i}_{j < k} \) determines either \( d^{i}_{j} \) or \( d^{i}_{k} \). The situation in the case where \( W_{j} \) is an antichain and \( W_{k} \) is increasing is similar. Table 2 summarizes the interactions between \( W_{j} \) and \( W_{k} \) in \( \mathcal{T} \) based on this information. The first eight columns denote whether key states are accepting (the value 1 denotes membership in \( F \); 0 denotes nonmembership in \( F \)). The next column gives a representative diagram of the \( <_{T} \) order of typical elements in \( W_{j} \) and \( W_{k} \).

**Lemma 6.6.** Any unary automatic tree is isomorphic to the tree-unfolding \( UF(\Gamma) \) of some parameter set \( \Gamma = (T_{0}, T_{1}, \ldots, T_{m}, \sigma, X) \).

**Proof.** Let \( \mathcal{T} = (\mathcal{N}; <_{T}) \) be a tree recognized by a standard unary automaton \( \mathcal{A} = (S, I, \Delta, F) \) with loop constant \( p \). The set \{\( y : y < p \}\) is a forest under \( <_{T} \). We define an equivalence relation \( \sim \) on \{\( y : y < p \}\) by \( x \sim y \) if and only if there are \( j, k \) such that \( x \in W_{j}, y \in W_{k} \) and \( W_{j}, W_{k} \) are not incomparable (see Table 2). There are finitely many \( \sim \)-equivalence classes \( M_{1}, \ldots, M_{r} \). Each \( M_{i} \) is a forest under \( <_{T} \). If \( i \not= i' \) and \( x \in M_{i}, y \in M_{i'} \), then \( x <_{T} y \).

The parameter set for \( \mathcal{T} \) has finite trees \( T_{0}, T_{1}, \ldots, T_{m} \). For \( i > 0 \), \( T_{i} \) is a subtree of \( M_{i} \) and a distinguished node \( x_{i} \) connects one copy of \( T_{i} \) to the root of the next copy. We extract the pairs \( (T_{i}, x_{i}) \) from \( \mathcal{A} \) as follows. For each \( 1 \leq i \leq s \), let \( C_{i} = \{ j : W_{j} \subseteq M_{i} \land W_{j} \text{ is a chain} \} \) and \( D_{i} = \{ j : W_{j} \subseteq M_{i} \land W_{j} \text{ is an anti-chain} \} \). The finite tree \( T_{i} \) has \( |C_{i}| + |D_{i}| \) many nodes, each representing a unique \( W_{j} \). The union of all nodes in the representative ordering of \( (W_{j}, W_{k}) \) for \( j, k \in C_{i} \) (from Table 2) forms a linear order under \( <_{T} \). Let \( c^{i}_{1} \) \( <_{T} \ldots \) \( <_{T} c^{i}_{|C_{i}|} \) be the \( |C_{i}| \)-many \( <_{T} \)-greatest nodes in this linear order, and set \( x_{i} = c^{i}_{|C_{i}|} \). Note that each \( c^{i}_{j} \) belongs to a different \( W_{j} \). For \( 1 \leq j \leq |D_{i}| \), let \( d_{j} \) be the \( <_{T} \)-least node in \( W_{j} \) satisfying \( c^{i}_{j} \) \( <_{T} d_{j} \). Define \( T_{0} \) to be the finite tree under \( <_{T} \) with domain \( \{ c^{i}_{1}, \ldots, c^{i}_{|C_{i}|} \cup \{ d_{1}, \ldots, d_{|D_{i}|} \} \} \). Then \( c^{i}_{1} \) is the root of \( T_{0} \). Let \( T_{0} \) be the finite tree formed by nodes in \{\( y : y < p \\} \cup \{ x : x \in M_{i} \land x <_{T} c^{i}_{j} \lor \forall y \in T_{0} (y <_{T} c^{i}_{j} \rightarrow y <_{T} x) \} \). Note that we must include the possibility that \( x_{T} c^{i}_{j} \); for example, in the seventh line of Table 2, \( v^{0} \) will be incomparable to the root of \( c^{i}_{j} \) (where \( W_{k} \subseteq M_{i} \)). \( T_{0} \) may be computed by examining whether \( (0, 1) \) -and \( (1, 0) \)-states are accepting and by using the case analysis in Table 2. To conclude the definition of \( \Gamma \), for \( 1 \leq i < s \), set \( \sigma(i) = x \) such that \( x \in T_{0} \) and \( x <_{T} c^{i}_{j} \) and \( \forall y \in T_{0} (y <_{T} c^{i}_{j} \rightarrow y <_{T} x) \). \( \Box \)

**Lemma 6.7.** If \( \Gamma = (T_{0}, T_{1}, \ldots, T_{m}, \sigma, X) \) is a parameter set, \( UF(\Gamma) \) is a unary automatic tree \( \mathcal{T} \).
Table 2: Relationship between $W_j$ and $W_k$ in tree $T$, based on $F$ in $\mathcal{A}$

<table>
<thead>
<tr>
<th>$q_j^m$</th>
<th>$q_k^n$</th>
<th>$q_{j&lt;k}$</th>
<th>$q_{j&gt;k}$</th>
<th>$q_{k&lt;j}$</th>
<th>$q_{k&gt;j}$</th>
<th>Ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$v_0 \rightarrow v_1 \rightarrow v_2 \cdots v_i \rightarrow v_{i+2} \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$v_0 \rightarrow v_1 \rightarrow v_2 \cdots v_i \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$v_0 \rightarrow v_0 \rightarrow v_2 \cdots v_i \rightarrow v_i \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>incomparable</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$v_0 \rightarrow v_1 \rightarrow v_2 \cdots v_i \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$v_0 \rightarrow v_0 \rightarrow v_1 \rightarrow v_2 \cdots v_i \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>incomparable</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$v_0 \rightarrow v_0 \rightarrow v_1 \rightarrow v_2 \cdots v_i \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$v_0 \rightarrow v_1 \rightarrow v_2 \cdots v_i \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$v_0 \rightarrow v_0 \rightarrow v_1 \rightarrow v_2 \cdots v_i \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>incomparable</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$v_0 \rightarrow v_0 \rightarrow v_1 \rightarrow v_2 \cdots v_i \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$v_0 \rightarrow v_0 \rightarrow v_1 \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$v_0 \rightarrow v_0 \rightarrow v_1 \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$v_0 \rightarrow v_0 \rightarrow v_1 \rightarrow v_{i+1}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>incomparable</td>
</tr>
</tbody>
</table>
Corollary 6.8. If \( T \) is recognized by a standard automaton with \( n \) states, an \( O(n) \) algorithm gives a parameter set \( \Gamma \) where \( T = \text{UF}(\Gamma) \).

Proof. The construction outlined in the proof of Lemma 6.6 uses finitely many table lookups in Table 2 and a single traversal of the transition diagram of the automaton recognizing \( T \).

Corollary 6.9. If \( \Gamma \) is a parameter set with \( t = |T_0| \) and \( \ell = \sum_{r=1}^{m} |T_r| \) then there is a standard unary automaton \( \mathcal{A} \) with \( O(t^2\ell^2) \) states such that \( \text{UF}(\Gamma) \equiv (\mathbb{N}; L(\mathcal{A})) \).

We now show that the isomorphism problem for unary automatic trees is decidable. Observe that two tree-unfoldings may be isomorphic even if the associated parameter sets are not isomorphic term-by-term. Ideally, we are looking for an isomorphism invariant which does not have this flaw. To obtain one, we begin by fixing a computable linear order \(<\) on the set of finite trees. We assume that the finite trees can be efficiently encoded as natural numbers such that asking if one is \(<\)-below another takes constant time. We define the canonical representation of a unary automatic tree \( T = (\mathbb{N}; \leq_T) \) to be the minimal parameter set \( \Gamma = (T_0, T_1, \ldots, T_m, \sigma, X) \) with \( \text{UF}(\Gamma) \equiv T \), where minimality is defined as follows.

- As finite trees, \( T_1 \leq \cdots \leq T_m \).
- Each \( T_i \) (1 \leq i \leq m) is minimal in that, for all \( y_1, y_2 \), if \( y_1 \leq_T y_2 \) \( x_i \), then the subtree with domain \( \{z : y_1 \leq_T z, y_2 \not\leq_T z\} \) is not isomorphic to the subtree with domain \( \{z : y_2 \leq_T z, x_i \not\leq_T z\} \). Also, if \( t_i \) is the root of the first copy of \( T_i \) (1 \leq i \leq m) then there is no \( y \in T_0 \) such that \( y <_T \sigma(i) \) and the subtree with domain \( \{z : y \leq_T z, t_i \not\leq_T z\} \) is isomorphic to \( T_i \).
Lemma 6.10. Suppose $\mathcal{T}, \mathcal{T}'$ are unary automatic trees with canonical representations $\Gamma, \Gamma'$. Then, $\mathcal{T} \cong \mathcal{T}'$ if and only if $\Gamma, \Gamma'$ have the same number $(m)$ of finite trees, $(T_0, \sigma) \cong (T'_0, \sigma')$, and for $1 \leq i \leq m$, $(T_i, x_i) \cong (T'_i, x'_i)$.

Proof. It is easy to see that if $\mathcal{T}$ and $\mathcal{T}'$ have term-by-term isomorphic canonical representations they are isomorphic. Conversely, suppose $\mathcal{T} \cong \mathcal{T}'$ and have canonical representations $(T_0, ..., T_m, \sigma, X)$ and $(T'_0, ..., T'_m, \sigma', X')$, respectively. Each infinite subtree $T_i$ of the form $\{(y : \sigma(i) \leq y) : \leq_T\}$, $1 \leq i \leq m$, which contains infinitely many copies of $T_i$, embeds into a subtree of $\mathcal{T}'$. By the minimality condition on $T_i$, $T'_i$ and by the ordering of the finite trees in each parameter set, the subtree of $\mathcal{T}$ containing infinitely many copies of $T_i$ can embed into the subtree of $\mathcal{T}'$ containing infinitely many copies of $T'_i$ for all $1 \leq i \leq m$ and vice versa. By minimality of $T_0, T'_0$, $\forall 1 \leq i \leq m$ $(T_i, x_i) \cong (T'_i, x'_i)$. Let $t_i$ be the root of the first copy of $T_i$ in $\mathcal{T}$, let $t'_i$ be the root of the first copy of $T'_i$ in $\mathcal{T}'$.

$$(T_0, \sigma) \cong ((y : y \in T_0 \land \forall 1 \leq i \leq m \sim t_i \leq_T y) ; \leq_T)$$

$$(T'_0, \sigma') \cong ((y : y \in T'_0 \land \forall 1 \leq i \leq m \sim t'_i \leq_T y) ; \leq_{T'})$$

Suppose we can compute the canonical representation of a tree from a unary automaton. Given two unary automatic trees, we could use Lemma 6.10 and a decision procedure for isomorphism of finite trees to solve the isomorphism problem on unary automatic trees.

Lemma 6.11. Given a tree-unfolding $\text{UF}(\Gamma)$ with $n$ the sum of the sizes of all finite trees in $\Gamma$, there is an $O(n^2)$ algorithm that computes the canonical representation of $\text{UF}(\Gamma)$.

Proof. Suppose $\Gamma = (T_0, T_1, ..., T_m, \sigma, X)$. For each $1 \leq i \leq m$, look for $y_1, y_2 \in T_i$ such that $y_1 \lesssim T y_2 \lesssim T x_i$, and the subtree of $T_i$ with domain $\{z : y_1 \leq_T z, y_2 \nleq_T z\}$ is isomorphic to the subtree with domain $\{z : y_2 \leq_T z, x_i \nleq_T z\}$. If such $y_1, y_2$ exist, remove all $z \geq_T y_1$ from $T_i$. Thus, each $T_i$ satisfies the minimality condition for the canonical representation. Since the isomorphism problem for finite trees is decidable in linear time [18], this step can be done in time $O(m[T_i]^2)$.

For each $1 \leq i \leq m$, let $t_i$ be the root of the first copy of $T_i$. Look for $x \in T_0$ such that $x \lesssim T \sigma(i)$, and the subtree of $T_0$ with domain $\{y : x \leq_T y, t_i \nleq_T y\}$ is isomorphic to $T_i$. If such an $x$ exists, remove all $y \geq_T x$ from $T_0$. Now $T_0$ satisfies the minimality condition. This step can be done in time $O(m[T_i]^2)$.

For each $1 \leq i \leq m$, search for the $<_{T_i}$-least $x$ such that the subtree of $T_0$ with domain $\{z \in T_0 : x \leq_{T_0} z\}$ is isomorphic to a subtree of $T_i$ with domain $\{z \in T_i : y \leq_{T_i} z\}$ for some $y \lesssim_{T_i} x$. If such an $x$ exists, remove all $y \geq_{T_0} x$ from $T_0$. This step ensures that $T_0$ has the fewest possible nodes with respect to $T_i$; it can be done in time $O(m[T_i]^2)$.

The last step in transforming our parameter set to the canonical presentation is to order the finite trees in increasing $\prec$-order. By assumption on the complexity of $\prec$,
Table 3: Summary of Results on Unary Automatic Structures

<table>
<thead>
<tr>
<th>Problems</th>
<th>Equivalence Structures</th>
<th>Linear Orders</th>
<th>Trees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Membership Problem</td>
<td>$O(n^3)$</td>
<td>$O(n^3)$</td>
<td>$O(n^4)$</td>
</tr>
<tr>
<td>Isomorphism Problem</td>
<td>$O(\max{n_1, n_2})$</td>
<td>$O(\max{n_1, n_2})$</td>
<td>$O(\max{n_1^2, n_2^2})$</td>
</tr>
</tbody>
</table>

applying a sorting algorithm on $m$ finite trees takes $O(m \log m)$. Since $n = |T_1| + \ldots + |T_m|$, the algorithm takes time $O(n^2)$. □

**Theorem 6.12.** If $T_1, T_2$ are unary automatic trees presented by standard automata $A_1 (n_1 \text{ states})$ and $A_2 (n_2 \text{ states})$, an $O(\max\{n_1^2, n_2^2\})$-time algorithm decides if $T_1 \equiv T_2$.

**Proof.** By Corollary 6.8 and Lemma 6.11, we can convert the standard automata presenting $T_1$ and $T_2$ to canonical representations of the trees. Then, the isomorphism problem reduces to checking finitely many isomorphisms of finite trees. The sum of sizes of finite trees in each parameter set is bounded by $n_i$. Hence, it takes $O(n_i^2)$ to compute each canonical representation and then check if they are equal. □

### 7. Conclusion and Future Work

We described algorithms deciding the isomorphism problems for unary automatic equivalence structures, linear orders, and trees. This settled the question of whether such algorithms existed. Moreover, we considered a normal form for the automata involved, with respect to which the time-complexity of the algorithms was polynomial. The membership problem for each of these classes was also shown to take polynomial-time with respect to any input unary automaton. It is still open whether the isomorphism problem for unary automatic graphs is decidable, and if so, what complexity class it lies in.

### References


