QUASICONTINUOUS SELECTIONS OF
UPPER CONTINUOUS SET-VALUED
MAPPINGS

Abstract
In this paper, we extend a theorem of Matejdes on quasicontinuous selections of upper Baire continuous set-valued mappings from compact (or separable) metric range spaces to regular $T_1$ range spaces. In addition, we also prove a quasicontinuous selection theorem for a special class of upper semicontinuous set-valued mappings.

1 Introduction.
Let $T : X \to 2^Y$ be a set-valued mapping with non-empty values. By a selection $f$ of $T$, we mean a single-valued mapping $f : X \to Y$ such that $f(x) \in T(x)$ for all $x \in X$. A well-known theorem of Michael on selections in [8] claims that any lower semicontinuous set-valued mapping $T : X \to 2^Y$ with non-empty closed convex values acting from a paracompact space $X$ into a Banach space $Y$ has a continuous selection. However, the conclusion of this theorem fails when lower semicontinuity is replaced by upper semicontinuity. For example, the set-valued mapping $T : \mathbb{R} \to 2^\mathbb{R}$, defined by

$$T(x) := \begin{cases} \{1/x\} & \text{if } x \neq 0 \\ \mathbb{R} & \text{if } x = 0 \end{cases}$$
is upper semicontinuous with non-empty closed convex values. Note that this mapping does not even possess a quasicontinuous selection. Recall that a (single-valued) mapping \( f : X \to Y \) is *quasicontinuous* if for every pair of open sets \( U \subseteq X \) and \( W \subseteq Y \) with \( f(U) \cap W \neq \emptyset \), there exists a non-empty open set \( V \subseteq U \) such that \( f(V) \subseteq W \). In a series of papers \([4, 5, 6, 7]\), Matejdes studied the problem of when a set-valued mapping admits a quasicontinuous selection. To achieve his goal, Matejdes introduced the following definition, \([4]\).

**Definition 1.1** (\([4]\)). A set-valued mapping \( T : X \to 2^Y \) is called *upper Baire continuous* at a point \( x \in X \) if for each pair of open sets \( U \) and \( W \) with \( x \in U \) and \( T(x) \subseteq W \), there is a subset \( B \subseteq U \) of the second category, having the Baire property, such that \( T(z) \subseteq W \) for all \( z \in B \).

We shall say that a set-valued mapping \( T : X \to 2^Y \) is *upper Baire continuous* if it is upper Baire continuous at every point of \( X \), and a Baire continuous single-valued mapping is just a special case of an upper Baire continuous set-valued mapping. Analogously, one can define lower Baire continuity for a set-valued mapping. However, we shall not do so here, since we are not going to use such a notion in this paper.

The following two facts on (upper) Baire continuity of mappings can be readily proved:

- If \( f : X \to 2^Y \) is upper Baire continuous with non-empty values, then \( X \) is Baire.
- If a (single-valued) mapping \( f : X \to Y \) is Baire continuous, \( X \) is Baire and \( Y \) is regular, then \( f \) must be quasicontinuous, \([4]\).

Using the previous two facts Matejdes proved the following theorem.

**Theorem 1.2** (\([4]\)). Let \( X \) be a \( T_1 \)-space and \( Y \) be a compact metric space. If \( T : X \to 2^Y \) is upper Baire continuous with non-empty compact values, then \( T \) admits a quasicontinuous selection.

In \([5]\), it was further shown that the compactness of \( Y \) in the previous theorem can be relaxed to the separability of \( Y \). The main purpose of this paper is to extend Theorem 1.2 using a different approach. Specifically, in Section 2, we show that the conclusion of Theorem 1.2 still holds when the condition “\( Y \) be a compact (or separable) metric space” is weakened to “\( Y \) be a regular \( T_1 \)-space”. The last section is dedicated to the study of quasicontinuous selections of a special class of upper semicontinuous set-valued mappings. Throughout the paper, \( T : X \to 2^Y \) always denotes a set-valued mapping acting from a topological space \( X \) to a topological space \( Y \) and \( f : X \to Y \) stands for a
single-valued mapping from $X$ into $Y$. The graph $\text{Gr}(T)$ of $T : X \to 2^Y$ is defined by

$$\text{Gr}(T) := \{(x, y) \in X \times Y : y \in T(x)\}.$$ 

All of our notation is standard and any undefined concepts may be found in the references.

2 An Extension of Theorem 1.2.

Let $X$ be a topological space. Recall that a set $A \subseteq X$ is said to be residual if $X \setminus A$ is a set of first category. As usual, the symmetric difference of two sets $A$ and $B$ in $X$ is denoted by $A \Delta B$. A set $A \subseteq X$ is said to have the Baire property if $A \Delta G$ is a set of the first category for some open set $G \subseteq X$.

The following characterization for upper Baire continuity of a set-valued mapping is easier to work with than the original definition in Definition 1.1.

**Lemma 2.1.** A set-valued mapping $T : X \to 2^Y$ with non-empty values is upper Baire continuous if, and only if, $X$ is Baire and for each pair of open subsets $U$ and $W$ with $x \in U$ and $T(x) \subseteq W$, there exist a non-empty open set $V \subseteq U$ and a residual set $R \subseteq V$ such that $T(z) \subseteq W$ for all $z \in R$.

**Proof.** ($\Rightarrow$). Suppose that $T : X \to 2^Y$ is upper Baire continuous. First, by remarks in Section 1, $X$ must be Baire. Furthermore, by the definition, for each pair of open sets $U$ and $W$ with $x \in U$ and $T(x) \subseteq W$, there exists some subset $B \subseteq U$ of the second category having the Baire property such that $T(z) \subseteq W$ for all $z \in B$. Let $B = G \Delta C$, where $G$ is an open set and $C$ is a set of the first category. Next, put $V = G \cap U$ and $R = G \setminus C$. Then $V \subseteq U$ is a non-empty open set and $R$ is a residual set in $V$ such that $T(z) \subseteq W$ for each $z \in R$.

($\Leftarrow$). Conversely, suppose that $X$ is Baire and for each pair of open sets $U$ and $W$ with $x \in U$ and $T(x) \subseteq W$, there exists a non-empty open subset $V \subseteq U$ and a residual subset $R \subseteq V$ such that $T(z) \subseteq W$ for all $z \in R$. Since $V$ is of the second category, then $R$ must be of the second category. In addition, $R = V \Delta (V \setminus R)$. Thus, $R$ has the Baire property as well.

Our next theorem extends Theorem 1.2 from a compact (or separable) metric range space to an arbitrary regular $T_1$ range space.

**Theorem 2.2.** Let $X$ be a topological space and $Y$ be a regular $T_1$-space. If $T : X \to 2^Y$ is an upper Baire continuous set-valued mapping with non-empty compact values, then $T$ admits a quasicontinuous selection.
Proof. First, by Lemma 2.1, \( X \) must be a Baire space. Let \( \mathcal{M} \) be the family of all upper Baire continuous set-valued mappings from \( X \) to \( Y \) with non-empty compact values such that for every \( H \in \mathcal{M}, \text{Gr}(H) \subseteq \text{Gr}(T) \). Since \( T \in \mathcal{M}, \mathcal{M} \neq \emptyset \). We define a partial order \( \preceq \) on \( \mathcal{M} \) by writing

\[
H_1 \preceq H_2 \text{ if, and only if, } \text{Gr}(H_1) \subseteq \text{Gr}(H_2).
\]

Next, we show that \( \mathcal{M} \) has a minimal element. To this end, let \( \mathcal{M}_0 \) be any linearly ordered non-empty subfamily of \( \mathcal{M} \). Then, define a set-valued mapping \( H_{\mathcal{M}_0} : X \to 2^Y \) by letting

\[
H_{\mathcal{M}_0}(x) := \bigcap \{H(x) : H \in \mathcal{M}_0\}
\]

for all \( x \in X \). Fix an arbitrary point \( x_0 \in X \). Since \( \{H(x_0) : H \in \mathcal{M}_0\} \) is a linearly ordered family of non-empty compact subsets of \( Y \), \( H_{\mathcal{M}_0}(x_0) \) is also a non-empty compact subset of \( Y \). Now, suppose that \( U \subseteq X \) and \( W \subseteq Y \) are a pair of non-empty open subsets with \( x_0 \in U \) and \( H_{\mathcal{M}_0}(x_0) \subseteq W \). Then, there must be some element \( H \in \mathcal{M}_0 \) such that \( H(x_0) \subseteq W \). By upper Baire continuity of \( H \) at \( x_0 \), there is a non-empty open set \( V \subseteq U \) and a residual subset \( R \subseteq V \) such that \( H(x) \subseteq W \) for all \( x \in R \). This implies that \( H_{\mathcal{M}_0}(x) \subseteq W \) for all \( x \in R \). Thus, \( H_{\mathcal{M}_0} \in \mathcal{M} \). By Zorn’s lemma, \( \mathcal{M} \) has a minimal member, which we will denote by \( \Phi_{\mathcal{M}} \).

Claim 1. For each pair of open subsets \( U \subseteq X \) and \( W \subseteq Y \) such that \( \Phi_{\mathcal{M}}(U) \cap W \neq \emptyset \), there exist a non-empty open subset \( V \subseteq U \) and a residual set \( R \subseteq V \) such that \( \Phi_{\mathcal{M}}(x) \subseteq W \) for all \( x \in R \).

Proof. Suppose the contrary. Then, there is a pair of open subsets \( U \subseteq X \) and \( W \subseteq Y \) with \( \Phi_{\mathcal{M}}(U) \cap W \neq \emptyset \) such that for every non-empty open subset \( V \subseteq U \) and every residual subset \( R \subseteq V \) there exists an \( x \in R \) such that \( \Phi_{\mathcal{M}}(x) \not\subseteq W \). Since \( \Phi_{\mathcal{M}} \) is upper Baire continuous, this implies that \( \Phi_{\mathcal{M}}(x) \not\subseteq W \) for any \( x \in U \). Next, we define a set-valued mapping \( \Gamma : X \to 2^Y \) by

\[
\Gamma(x) := \begin{cases} 
\Phi_{\mathcal{M}}(x) \cap \{Y \setminus W\} & \text{if } x \in U \\
\Phi_{\mathcal{M}}(x) & \text{otherwise.}
\end{cases}
\]

Then \( \Gamma \) has non-empty compact values. We will show that \( \Gamma \) is upper Baire continuous. Pick any point \( x_0 \in X \). If \( x_0 \not\in U \), then the result is clear, since \( \Phi_{\mathcal{M}} \) is upper Baire continuous and \( \Gamma \preceq \Phi_{\mathcal{M}} \). Assume \( x_0 \in U \). Let \( U' \) and \( W' \) be a pair of open sets with \( x_0 \in U' \subseteq U \) and \( \Gamma(x_0) \subseteq W' \). Then \( \Phi_{\mathcal{M}}(x_0) \subseteq W \cup W' \). Thus there exist a non-empty open set \( V' \subseteq U' \) and a residual set \( R' \subseteq V' \) such that \( \Phi_{\mathcal{M}}(x) \subseteq W \cup W' \) for all \( x \in R' \). Clearly, \( \Gamma(x) \subseteq W' \) for every point \( x \in R' \). This implies that \( \Gamma \) is upper Baire.
continuous at every point of $U$. Thus, we have shown that $\Gamma \in \mathcal{M}$. But this is impossible since $\Gamma \preceq \Phi_M$ and $\Phi \neq \Phi_M$. Hence we have obtained our desired contradiction.

**Claim 2.** $\Phi_M$ is single-valued at every point $x \in X$.

**Proof.** If not, there must exist a point $x_1 \in X$ such that $\Phi_M(x_1)$ contains at least two points. Now, pick any point $y_1 \in \Phi_M(x_1)$, and then define another set-valued mapping $\Psi : X \to 2^Y$ by

$$
\Psi(x) := \begin{cases} 
\{y_1\}, & \text{if } x = x_1, \\
\Phi_M(x), & \text{otherwise}.
\end{cases}
$$

It is clear that $\Psi$ has non-empty compact images. Let $x \in X$ and consider open sets $U \subseteq X$ and $W \subseteq Y$ such that $x \in U$ and $\Psi(x) \subseteq W$. By Claim 1, there exist a non-empty open subset $V \subseteq U$ and a residual subset $R \subseteq V$ such that $\Phi_M(x) \subseteq W$ for all $x \in R$. It follows that $\Psi(x) \subseteq W$ for all $x \in R$. Thus $\Psi$ is upper Baire continuous. But, $\Psi \preceq \Phi_M$ and $\Psi \neq \Phi_M$; which contradicts the minimality of $\Phi_M$.

Finally, by Claim 2, $\Phi_M$ is a Baire continuous selection of $T$. Therefore, since $X$ is Baire and $Y$ is regular, $\Phi_M$ is quasicontinuous. \hfill \Box

### 3 Strongly Injective Set-Valued Mappings

In this section, we shall examine when an upper semicontinuous set-valued mapping acting between topological spaces admits a quasicontinuous selection. Recall that a set-valued mapping $T : X \to Y$ from a topological space $X$ into a topological space $Y$ is said to be upper semicontinuous at a point $x_0 \in X$ if for every open subset $V \subseteq Y$ with $T(x_0) \subseteq V$, there exists an open subset $U \subseteq X$ with $x_0 \in U$ such that $T(U) \subseteq V$.

Our considerations are based upon the following notion.

**Definition 3.1.** A set-valued mapping $T : X \to 2^Y$ is strongly injective if $T(x_1) \cap T(x_2) = \emptyset$ for any two distinct points $x_1, x_2 \in X$.

**Remark 3.2.** If $f : X \to Y$ is a surjective mapping, then $f^{-1} : Y \to 2^X$ is strongly injective. In particular, the quotient mapping $q : G \to G/H$ from a (Hausdorff) group $G$ onto a coset space $G/H$ as considered by Michael in [8] is strongly injective. Conversely, for any strongly injective set-valued mapping $T : Y \to 2^X$ with non-empty values and $T(Y) = X$, it is easy to see that there exists a mapping $f : X \to Y$ such that $T = f^{-1}$. 
Furthermore, we shall also require the definition of property \((**\)) introduced in [2]. Let \(X\) be a space, \(\mathcal{F}\) a proper filter (or filterbase) in \(X\). We shall consider the following \(G(\mathcal{F})\)-game played in \(X\) between players \(A\) and \(B\): Player \(A\) goes first (always!) and chooses a point \(x_1 \in X\). Player \(B\) responds by choosing a member \(F_1 \in \mathcal{F}\). Following this, player \(A\) must select another (possibly the same) point \(x_2 \in F_1\) and in turn player \(B\) must again respond to this by choosing a member \(F_2 \in \mathcal{F}\). Repeating this procedure indefinitely, the players \(A\) and \(B\) produce a sequence \(p = ((x_n, F_n) : n \in \mathbb{N})\) with \(x_{n+1} \in F_n\) for all \(n \in \mathbb{N}\), called a play of the \(G(\mathcal{F})\)-game. We shall say that \(B\) wins a play of the \(G(\mathcal{F})\)-game if the sequence \((x_n : n \in \mathbb{N})\) has a cluster point in \(X\). Otherwise, the player \(A\) is said to have won this play.

We shall call a pair \((\mathcal{F}, \sigma)\) a \(\sigma\)-filter (\(\sigma\)-filterbase) if \(\mathcal{F}\) is a proper filter (filterbase) in \(X\) and \(\sigma\) is a winning strategy for player \(B\) in the \(G(\mathcal{F})\)-game. Finally, we say that a space \(X\) has property \((**\)) if \(\bigcap\{F : F \in \mathcal{F}\} \neq \emptyset\) for each \(\sigma\)-filterbase \((\mathcal{F}, \sigma)\) in \(X\). The class of spaces having property \((**\)) includes all metric spaces [1], all Dieudonné-complete spaces, all function spaces \(C_p(X)\) for compact Hausdorff spaces \(X\), and all Banach spaces in their weak topologies [2].

The following theorem may be deduced from [2, Theorem 3.3].

**Theorem 3.3** ([2]). Let \(T : X \rightarrow 2^Y\) be a strongly injective upper semicontinuous set-valued mapping with non-empty closed values. If \(X\) is a regular \(q\)-space and \(Y\) is a regular space with property \((**\))\), then for any point \(x_0 \in X\),

\[
K := \bigcap_{U \in \mathcal{U}(x_0)} \overline{T(U \setminus \{x_0\})}
\]

is a compact subset of \(T(x_0)\), where \(\mathcal{U}(x_0)\) is the family of all neighborhoods of \(x_0\) in \(X\) and \(\overline{T(U \setminus \{x_0\})}\) is the closure of \(T(U \setminus \{x_0\})\) in \(Y\). In addition, the mapping \(T_K : X \rightarrow 2^Y\), defined by

\[
T_K(x) := \begin{cases} 
K & \text{if } x = x_0, \\
T(x) & \text{otherwise},
\end{cases}
\]

is upper semicontinuous on \(X\).

Note that, in the previous theorem, if \(x_0 \in X\) is not an isolated point, then \(K\) is non-empty.

Our next selection theorem requires the notion of a minimal usco.
Definition 3.4. We shall call a set-valued mapping \( \varphi : X \to 2^Y \) acting between topological spaces \( X \) and \( Y \) an \textit{usco} mapping if for each \( x \in X \), \( \varphi(x) \) is a nonempty compact subset of \( Y \) and for each open set \( W \) in \( Y \) \{ \( x \in X : \varphi(x) \subseteq W \) \} is open in \( X \). An usco mapping \( \varphi : X \to 2^Y \) is called a \textit{minimal} usco if its graph does not contain, as a proper subset, the graph of any other usco defined on \( X \).

Proposition 3.5 ([3]). Let \( \varphi : X \to 2^Y \) be an usco acting between topological spaces \( X \) and \( Y \). Then \( \varphi \) is a minimal usco if and only if, for each pair of open subsets \( U \) of \( X \) and \( W \) of \( Y \) with \( \varphi(U) \cap W \neq \emptyset \) there exists a non-empty open subset \( V \) of \( U \) such that \( \varphi(V) \subseteq W \). In particular, every selection of a minimal usco is quasicontinuous.

Proposition 3.6 ([3]). Let \( \varphi : X \to 2^Y \) be an usco mapping acting from a topological space \( X \) into a Hausdorff topological space \( Y \). Then there exists a minimal usco \( \psi : X \to 2^Y \) such that \( \psi(x) \subseteq \varphi(x) \) for all \( x \in X \).

Theorem 3.7. Let \( T : X \to 2^Y \) be a strongly injective upper semicontinuous set-valued mapping with nonempty closed values. If \( X \) is a regular \( q \)-space and \( Y \) is a regular Hausdorff space with property \( (**) \), then \( T \) admits a quasicontinuous selection.

Proof. For any isolated point \( x \in X \), pick an arbitrary point \( y_x \in T(x) \). Next, define the set-valued mapping \( \Phi : X \to 2^Y \) by,

\[
\Phi(x) := \begin{cases} 
\bigcap_{U \in \Phi(x)} T(U \setminus \{x\}) & \text{if } x \text{ is not isolated} \\
\{y_x\} & \text{if } x \text{ is isolated.}
\end{cases}
\]

By Theorem 3.3 and the subsequent remark, \( \Phi \) has non-empty compact values.

Now, fix an arbitrary point \( x_0 \in X \). To show that \( \Phi \) is upper semicontinuous at \( x_0 \), we consider two possible cases. If \( x_0 \) is an isolated point of \( X \), then the upper semicontinuity of \( \Phi \) at \( x_0 \) is trivial. In the case that \( x_0 \) is non-isolated, it follows from the second part of Theorem 3.3. Thus, \( \Phi \) is an usco whose graph is contained in the graph of \( T \). By Proposition 3.6, there exists a minimal usco \( \psi : X \to 2^Y \) such that \( \psi(x) \subseteq \Phi(x) \subseteq T(x) \) for all \( x \in X \).

Now, by Proposition 3.5, \( \psi \) has a quasicontinuous selection \( \sigma : X \to Y \) which in turn is also a selection of \( T \). \( \square \)
Corollary 3.8. Let \( f : X \to Y \) be a closed mapping from a regular \( T_1 \)-space \( X \) with property (**) onto a regular \( q \)-space \( Y \). If \( f^{-1}(y) \) is closed for every \( y \in Y \), then there exists a quasicontinuous mapping \( \varphi : Y \to X \) such that \((f \circ \varphi)(y) = y\) for all \( y \in Y \).

**Proof.** Note that \( f^{-1} : Y \to 2^X \) is an upper semicontinuous strongly injective set-valued mapping with non-empty closed values. By applying Theorem 3.7, \( f^{-1} \) admits a quasicontinuous selection \( \varphi : Y \to X \). Evidently, \((f \circ \varphi)(y) = y\) for all \( y \in Y \). \( \square \)

Remark 3.9. By [2, Theorem 1.2] and an argument similar to that in Theorem 3.7, one can show the following: Let \( T : X \to 2^Y \) be an upper semicontinuous set-valued mapping from a first countable space \( X \) into a Hausdorff and angelic space \( Y \). If \( T \) is strongly injective, then it admits a quasicontinuous selection. As a consequence of this result, the condition “\( f^{-1}(y) \) is closed for every \( y \in Y \)” in Corollary 3.8 can be dropped when \( X \) is Hausdorff and angelic and \( Y \) is first countable; i.e., for any closed mapping \( f : X \to Y \) from a Hausdorff and angelic space \( X \) onto a first countable space \( Y \), there exists a quasicontinuous mapping \( \varphi : Y \to X \) such that \((f \circ \varphi)(y) = y\) for all \( y \in Y \).

*Note Added in Proof:* We should observe that the conclusion of Theorem 3.7 remains if we replace the condition “\( T \) is strongly injective” by the weaker hypothesis that “\( T \) is locally strongly injective”; i.e., for each \( x \in X \) there exists a neighborhood \( U \) of \( x \) such that \( T|_U \) is strongly injective on \( U \).

References


