Pricing Variance Swaps under Stochastic Volatility Model with Regime Switching - Discrete Observations Case

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Outline

- Background
  
  An analytical solution for pricing variance swaps based on the Heston (1993) stochastic volatility model with regime switching

- Examples and Discussions

- Concluding Remarks
The first generation model: Black-Scholes model

\[ dS = rSdt + \sigma SdB_t \]

Black-Scholes formula

\[ C_t = S_tN(d_1) - K \exp[-r(T-t)]N(d_2) \]

where

\[ d_1 = \frac{\ln S_t/K + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \]

\[ d_2 = d_1 - \sigma\sqrt{T - t} \]

It is incapable of generating “volatility smile”.

The implied volatility is calculated from the ASX/SPI200 index call options which will expire in one month. Data are obtained from Australia Stock Exchange, on Feb. 8, 2010. The ASX/SPI index is 4521 on that date.
The second generation of models.

- Stochastic volatility models (Heston 1993; Stein and Stein 1991)
- Jump diffusion models (Bakshi et al. 1997; Duffie et al. 2000)
- Local volatility surface models (Dupire B. 1994).
The third generation of models.

- Models incorporating regime switching.
- Levy jump models (CGMY);
- VG models;
Why Regime Switching?

- Economic reasons: business cycles.
- It is necessary to allow the key parameters of the model to respond to the general market movements.
Why Regime Switching?

- Vo (2009) found strong evidence of regime-switching in the market, and showed that the regime-switching stochastic volatility model does a better job in capturing major events affecting the market.
The applications of regime switching models in finance include

- asset allocation (Elliott & Van der Hoek 1997);
- short term rate model and bond evaluation (Elliott & Siu 2009);
- portfolio analysis (Zhou & Yin 2004; Honda 2003);
- pricing options (Guo & Zhang 2004);
- risk management (Elliott et al. 2008).
There is a little work on pricing variance swaps in the context of regime-switching models.

- The only paper so far is Elliott et al. (2007).
- Their work for variance swaps is based on continuous observations in calculating realized variance.
- They have also pointed out that in practice, variance swaps are always written on the realized variance evaluated by a discrete summation based on daily closing prices, instead of a continuous observations.
**Background**

**What is a variance swap?**

A variance swap is a forward contract on the future realized variance of the underlying asset.

- Cash flow of a variance swap at expiration

\[ \sigma^2_R \] is the annualized realized variance over the contract life \( T \);

\[ K_{\text{Var}} \] is the annualized strike price for the variance swap.
The payoff of a variance swap at maturity $T$ is usually of the form:

$$V_T = (\sigma^2_R - K_{var}) \times L,$$

and $L$ is the notional amount of the swap per annualized volatility point squared, which is usually set to 10000.
Background

There are several different forms of $\sigma^2_R$:

\begin{equation}
\sigma^2_R = \frac{AF}{N} \sum_{k=1}^{N} \left( \frac{S_{tk} - S_{tk-1}}{S_{tk-1}} \right)^2
\end{equation}

or

\begin{equation}
\sigma^2_R = \frac{AF}{N} \sum_{k=1}^{N} \left[ \ln(S_{tk}) - \ln(S_{tk-1}) \right]
\end{equation}

or

\begin{equation}
\sigma^2_R = \frac{1}{T} \int_0^T v_t dt
\end{equation}
Background

- **Analytical Approaches:**

  - Carr and Madan (1998), Demeterfi et al. (1999): replicate a variance swap by a portfolio of options;
  - Heston (2000): analytical solution based on GARCH model;

  The limitation of these methods is the assumption that sampling frequency is high enough to allow the realized variance to be approximated by a continuously-sampled variance defined as

  \[
  \sigma^2_R = \frac{1}{T} \int_0^T v_t dt
  \]  

  (4)
Numerical Approaches:


- Windcliff et al. (2006): Integral differential equation approach for discretely sampled realized variance;

The drawback of these numerical approaches is that they are limited to the case with local volatility being a given function of the underlying asset and time.
Background

*Most Recent Research:*

To properly address the discretely sampling effect, several works have been completed, based on the Heston stochastic volatility model (SV)

- Broadie & Jain (2008);
- Itkin & Carr (2010);
- Zhu & Lian (2010);
The contributions of this study

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Our Closed-form Analytical Solution

Assumptions:
- Consider a continuous-time finite-state Markov chain $X = \{X_t\}_{t \in T}$

$$X_t = X_0 + \int_0^t AX_s ds + M_t,$$  \hspace{1cm} (5)

where $M_t$ is an martingale.

The finite-state space is identified with $S = \{e_1, e_2, ..., e_N\}$, where $e_i = (0, ..., 1, ..., 0) \in R^N$
Our Closed-form Analytical Solution

**Assumptions:**
- The realized variance is discretely sampled and defined as

\[ \sigma^2_R = \frac{A}{N} \sum_{i=1}^{N} \left( \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \]  

(6)

- The underlying asset and the instantaneous variance follow the dynamics:

\[ dS_t = r_t S_t dt + \sqrt{V_t} S_t dB^S_t, \]
\[ dV_t = \kappa (\theta - V_t) dt + \sigma V \sqrt{V_t} dB^V_t, \]  

(7)

respectively.

Our Closed-form Analytical Solution

\[ dS_t = r_t S_t dt + \sqrt{V_t} S_t dB^S_t, \]
\[ dV_t = \kappa (\theta_t - V_t) dt + \sigma_V \sqrt{V_t} dB^V_t. \]

- Here \( r \) is the risk-free interest rate, \( \theta \) is the long-term mean of the variance, \( \kappa \) is a mean-reverting speed parameter of the variance, \( \sigma_V \) is the so-called volatility of volatility.

\[ r_t = r(t, X_t) = < r, X_t >, \quad r = (r_1, r_2, ..., r_N) \]
\[ \theta_t = \theta(t, X_t) = < \theta, X_t >, \quad \theta = (\theta_1, \theta_2, ..., \theta_N) \]

- \( dB^S_t \) and \( dB^V_t \) are two Wiener processes that are correlated by a constant correlation coefficient \( \rho \), that is \( < B^S_t, B^V_t > = \rho t \).
Clearly, to calculate the price of an existing variance swap with a payoff $V_T = (\sigma_R^2 - K_{var}) \times L$ or to set up a strike price $K_{var}$ for a new contract, essentially, all one needs is to calculate the expectation of the unrealized variance:

$$K_{var} = E_Q^0 [\sigma_R^2] = E_Q^0 \left[ \frac{1}{T} \sum_{i=1}^{N} \left( \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right],$$

where $E_t^Q$ denotes the expectation under the $Q$ measure conditional on the information available at time $t$. 
If we further assume that the sampling points are equally spaced, i.e.,

$$AF = \frac{1}{\Delta t} = \frac{N}{T},$$

then

$$K_{var} = E_0^Q[\sigma_R^2] = E_0^Q\left[ \frac{1}{N\Delta t} \sum_{i=1}^{N} \left( \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right].$$

Thus, our problem essentially becomes to evaluate $N$ expectations

$$E_0^Q\left[ \left( \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \right]$$

(8)
Our Closed-form Analytical Solution

Characteristic Function Method:

- Assuming the current time is $t$, write $y_T = \log S_{T+\Delta} - \log S_T$.
- Define forward characteristic function $f(\phi; t, T, \Delta, V_t)$ of the stochastic variable $y_T$ as the Fourier transform of the probability density function of $y_T$, i.e.,

$$f(\phi; t, T, \Delta, V_t) = E_t^Q[e^{\phi y_T}]$$

$$= E_t^Q[\exp(\phi(\log S_{T+\Delta} - \log S_T))]$$

- Obtain this characteristic function and then solve the pricing of variance swaps.
Our Closed-form Analytical Solution

We combine the techniques of the tower rule (law of iterated expectation) and the partial differential equation (PDE).

- Step 1: conditional expectation.
  Given the filtration $F_{T+\Delta}^X$, the parameters $r_t$ and $\theta_t$ can be considered to be time-dependent deterministic functions.

- Step 2: characteristic function of regime switching process, $X_t$;
  Solve the PDE associated with the regime switching process;

- Step 3: unconditional expectation;
  Apply the results in step 1 and 2 to finally obtain the required characteristic function.

... mathematical derivations ...
Proposition 0.1

If the underlying asset follows the dynamics (7), then the forward characteristic function of the stochastic variable \( y_T = \log S_{T+\Delta} - \log S_T \) is given by:

\[
f(\phi; t, T, \Delta, V_t) = E^Q_t [e^{\phi y_T}]
\]

\[
= \exp \left( G(D(\phi, T), T - t)V_t \right) < \Phi(t, T)X_t, I >
\]

where \( D(\phi, t) \) is given by,

\[
\begin{align*}
D(\phi, t) &= \frac{a + b}{\sigma^2_V} \frac{1 - e^{b(T+\Delta-t)}}{1 - ge^{b(T+\Delta-t)}} \\
\begin{aligned}
a &= \kappa - \rho \sigma_V \phi, \quad &b &= \sqrt{a^2 + \sigma^2_V (\phi - \phi^2)} , \quad &g &= \frac{a + b}{a - b}
\end{aligned}
\end{align*}
\]
(Continue)

If the underlying asset follows the dynamics (7), then the forward characteristic function of the stochastic variable \( y_T = \log S_{T+\Delta} - \log S_T \) is given by:

\[
f(\phi; t, T, \Delta, V_t) = E_t^Q [e^{\phi y_T}]
\]

\[
= \exp \left( G(D(\phi, T), T - t)V_t \right) < \Phi(t, T)X_t, I >
\]

where \( G(\phi; t, T, V_t) \) is given by,

\[
G(\phi, t) = \frac{2\kappa \phi}{\sigma^2 V\phi + (2\kappa - \sigma^2 V\phi)e^{\kappa(T-t)}}
\]

\[
J(t) = (1 - H_T(t))(\kappa \theta G(D(\phi, T), t)) + H_T(t)(r\phi + \kappa \theta D(\phi, t))
\]

\[
\Phi(t, T) = \exp \left( \int_t^{T+\Delta} A' + \text{diag}(J(s))ds \right)
\]
Our Closed-form Analytical Solution

- Having worked out the forward characteristic function

\[ f(\phi; t, T, \Delta, V_t) = E_t^Q[e^{\phi y_T}] \]

- Pricing variance swaps becomes quite trivial.

\[ K_{var} = \frac{1}{T} \sum_{k=1}^{N} [f(2; 0, t_{k-1}, \Delta t, V_0) - 2f(1; 0, t_{k-1}, \Delta t, V_0) + 1] \]
Numerical Results

- Obtain numerical results from the implementation of our pricing formula.
- Monte Carlo benchmark values for testing purpose.
- Compare with the continuous sampling approximation.
Numerical Results

The model

\[ dS_t = rS_t dt + \sqrt{V_t}S_t dB^S_t, \]
\[ dV_t = \kappa(\theta - V_t) dt + \sigma_V \sqrt{V_t} dB^V_t, \]
\[ <B^S_t, B^S_t> = \rho t \]

\[ r_t = r(t, X_t) = <r, X_t>, \quad r = (r_1, r_2, \ldots, r_N) \]
\[ \theta_t = \theta(t, X_t) = <\theta, X_t>, \quad \theta = (\theta_1, \theta_2, \ldots, \theta_N) \]
\[ X_t = X_0 + \int_0^t AX_s ds + M_t, \]

Parameters \( \rho = -0.82; \ \kappa = 3.46; \)
\( \sigma_V = 0.14; \ V_0 = (8.7/100)^2; \)
\( A = [-0.1, 0.1; 0.4, -0.4]; \ X_0 = 1; \)
\( r = [0.06; 0.03]; \ \theta = [0.009; 0.004]. \)
MC simulations are frequently used, particularly when no closed-form solutions.

- obtain benchmark values for testing other methods.
- not feasible for practical use because of computational inefficiency.
We suggest a semi-MC method

Algorithm.
1. Let $N$ be the number of samplings. For each $n = 1, \ldots, N$, we then
2. obtain the $n$-th sampling path of the regime switching process, $X_T$;
3. with a realized sampling path of $X_T$, the characteristic function is presented in Proposition 1.

$$f(\phi; t, T, \Delta, V_t | F_{T+\Delta}^X) = E^Q[e^{\phi y_T} | F_t^S \lor F_t^V \lor F_{T+\Delta}^X]$$

$$= e^{C(\phi, T)} g(D(\phi, T); t, T, V_t)$$

So we can calculate the price of a variance swap for the $n$-th sampling path.
4. calculate the average $K = \frac{1}{N} \sum_{n=1}^{N} K_n$. 
The continuous observation case

- Elliott et al. (2007)’s formula

\[
P(X) = e^{-\int_0^T r_u du} N\left(\frac{\sigma_u^2}{\beta T} (1 - e^{-\beta T}) + \frac{\beta}{T} \int_0^T \left(\int_0^t \langle \bar{z}^2, X_t \rangle e^{-\beta(t-s)} ds \right) dt - K_v\right)
\]
Results and Discussions

- A comparison with the results obtained from other approaches:

![Graph showing comparison of results from different models]

- Results from our discrete observation model
- Results from the semi Monte Carlo simulation
- Results from continuous observation model (Elliot et al. 2007)
Results and Discussions

A comparison with the results obtained from other approaches:
An analytical solution is obtained for variance swaps based on a stochastic volatility model with regime switching;

For discretely sampled variances, it is more accurate to use our solution than using continuous approximations;

It examines the effect of ignoring regime switching on pricing variance and volatility swaps;

Our solution can be very efficiently computed; substantial computational time can be saved in comparison to Monte Carlo Method;
Thank you!

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